Lesson Plan for 11.4: Tensor Product of Two Vectors
MATH 564: Advanced Linear Algebra

Tatsuya Akiba

2017-11-30

1 Set-up

Recall: The inner product satisfies:

1) $(x, y) = \overline{(y, x)}$

2) $(x, x) \geq 0 \ \forall x$ and $(x, x) = 0$ iff $x = 0$

3) $(ax + by, z) = a(x, z) + b(y, z) \ \forall a, b \in \mathbb{F}$

In this case, the induced norm is given by $(x, x)^{1/2} \equiv |x|$

One important property:

$(x, ay + bz) = \overline{a}(x, y) + \overline{b}(x, z)$, because:

$$\begin{aligned}(x, ay + bz) &= \overline{(ay + bz, x)} \text{ by 1)} \\ &= \overline{a(y, x)} + \overline{b(z, x)} \text{ by 3)} \\ &= \overline{a}(x, y) + \overline{b}(x, z) \text{ again by 1)}\end{aligned}$$

Recall: A normed linear space in which the norm comes from an inner product as just described is called an inner product space.

Basically: An inner product space is a vector space with an additional structure called an inner product as defined in Definition 11.1.2.

Notation: We need to be able to distinguish between inner products in different inner product spaces, so the inner product of the inner product space X will be denoted by $(\cdot, \cdot)_X$.

2 Main Section

2.1 Definition and Basic Properties

Definition 11.4.1 Let X and Y be inner product spaces and let $u, x \in X$ and $y \in Y$. Define the tensor product of these two vectors, $y \otimes x \in \mathcal{L}(X, Y)$ by

$$(y \otimes x)(u) \equiv y(u, x)_X$$

Class Exercise: Prove that the tensor product is a linear map. (Give a few minutes)

Proof:

Let $u, x \in X$, $v, y \in Y$, and $a, b \in \mathbb{F}$.

$(y \otimes x)(au + bv)$

$= y(au + bv, x)_X$

$= y[a(u, x)_X + b(v, x)_X]$

$= ay(u, x)_X + by(v, x)_X$

$= a(y \otimes x)(u) + b(y \otimes x)(v)$

2.2 Specific Cases

2.2.1 Specific Case 1: $\mathbb{R}^n$

Definition: Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$. The outer product of these two coordinate vectors (in standard bases) is:

$$x \otimes y \equiv xy^T$$

$$= \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_n \\ x_2y_1 & x_2y_2 & \dots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \dots & x_ny_n \end{pmatrix}$$

Recall: Let $x, y \in \mathbb{R}^n$. In this case, the inner product (dot product) can be written as:

$$x \cdot y = x^T y$$

Claim: The outer product is the tensor product for vectors in $\mathbb{R}^n$ with the usual inner product above.

Proof:

$$(y \otimes x)(u) = (yx^T)(u)$$

$$= y(x^T u)$$

$$= y(x \cdot u)$$

$$= y(u \cdot x)$$

$$= y(u, x)_X$$

Example

Let $x = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $y = \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix}$, and $u = \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$. Compute the outer products $x \otimes y$ and $y \otimes x$, and verify that $(y \otimes x)(u) = y(u, x)_X$

$$x \otimes y = xy^T$$

$$= \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -4 & -2 \\ 3 & 0 & -6 & -3 \\ 1 & 0 & -2 & -1 \end{pmatrix}$$

$$y \otimes x = yx^T$$

$$= \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ -4 & -6 & -2 \\ -2 & -3 & -1 \end{pmatrix}$$

Notice: $x \otimes y \neq y \otimes x$ in general. In fact, in the specific case of $\mathbb{R}^n$ with the usual inner product, you'll notice that $x \otimes y = (y \otimes x)^T$.

$$(y \otimes x)(u)$$

$$= \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 0 \\ -4 & -6 & -2 \\ -2 & -3 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 0 \\ 20 \\ 10 \end{pmatrix}$$

$$y(u, x)_X$$

$$= \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix} \left[\begin{pmatrix} 0 \\ -4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 1 \\ 0 \\ -2 \\ -1 \end{pmatrix} (-10)$$

$$= \begin{pmatrix} -10 \\ 0 \\ 20 \\ 10 \end{pmatrix}$$

2.2.2 Specific Case 2: $\mathbb{C}^n$

Definition: Let $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$. The outer product of these two coordinate vectors (in standard bases) is:

$$x \otimes y \equiv x\overline{y}^T$$

Recall: Let $x, y \in \mathbb{C}^n$. In this case, the inner product can be written as:

$$x \cdot y = \overline{x}^T y$$

Example

Let $x = \begin{pmatrix} 2 \\ 3 \\ -i \end{pmatrix}$, $y = \begin{pmatrix} i \\ 0 \\ -2i \\ -1 \end{pmatrix}$. Compute the outer product $x \otimes y$.

$$x \otimes y = x\overline{y}^T$$

$$= \begin{pmatrix} 2 \\ 3 \\ -i \end{pmatrix} \begin{pmatrix} -i & 0 & 2i & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2i & 0 & 4i & -2 \\ -3i & 0 & 6i & -3 \\ -1 & 0 & 2 & i \end{pmatrix}$$

2.2.3 Specific Case 3: $\mathbb{R}^{m\times n}$ (the set of $m \times n$ real matrices)

Definition: Let $X \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{p\times q}$. The Kronecker product of these two matrices is the $mp \times nq$ block matrix given by:

$$X \otimes Y = \begin{pmatrix} x_{11}Y & \dots & x_{1n}Y \\ \vdots & \ddots & \vdots \\ x_{m1}Y & \dots & x_{mn}Y \end{pmatrix}$$

Example

Let $X = \begin{pmatrix} 1 & 3 \\ 0 & -2 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -3 \\ 2 & 1 \end{pmatrix}$. Compute the Kronecker product $X \otimes Y$.

$$X \otimes Y$$

$$= \begin{pmatrix} 1\begin{pmatrix} 0 & -3 \\ 2 & 1 \end{pmatrix} & 3\begin{pmatrix} 0 & -3 \\ 2 & 1 \end{pmatrix} \\ 0\begin{pmatrix} 0 & -3 \\ 2 & 1 \end{pmatrix} & -2\begin{pmatrix} 0 & -3 \\ 2 & 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -3 & 0 & -9 \\ 2 & 1 & 6 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & -4 & -2 \end{pmatrix}$$

2.3 Tensor Product Theory

Corollary 11.3.2: Let $A \in \mathcal{L}(X,Y)$ where X and Y are two inner product spaces of finite dimension. Then there exists a unique $A^* \in \mathcal{L}(Y,X)$ such that

$$(Ax, y)_Y = (x, A^*y)_X$$

$\forall\ x \in X$ and $y \in Y$.

Definition 11.3.3: The linear map, A^* is called the adjoint of A.

Lemma 11.4.2: Let X, Y, Z be inner product spaces and let $x \in X$, $y \in Y$, $z \in Z$ and α a scalar. Then:

1) $(\alpha(y \otimes x))^* = \overline{\alpha}(x \otimes y)$

2) $(z \otimes y_1)(y_2 \otimes x) = (y_2, y_1)(z \otimes x)$

Proof:

Let $u \in X$ and $v \in Y$.

1) By definition of the adjoint, we would like to show that

$(\alpha(y \otimes x)(u), v)_Y = (u, \overline{\alpha}(x \otimes y)(v))_X$

$LHS = (\alpha y(u,x)_X, v)_Y = \alpha(u,x)_X(y,v)_Y$

$RHS = (u, \overline{\alpha}x(v,y)_Y)_X = \alpha\overline{(v,y)_Y}(u,x)_X = \alpha(u,x)_X(y,v)_Y$

2) Show that both sides have the same effect on vector u:

$LHS = (z \otimes y_1)(y_2 \otimes x)(u)$

$= (z \otimes y_1)y_2(u,x)_X$

$= (u,x)_X z(y_2,y_1)_Y$

$= (u,x)_X(y_2,y_1)_Y z$

$RHS = (y_2,y_1)_Y(z \otimes x)(u)$

$= (y_2,y_1)_Y z(u,x)_X$

$= (u,x)_X(y_2,y_1)_Y z$

Recall Definition 11.1.10: A basis for an inner product space, $\{u_1, ..., u_n\}$ is an orthonormal basis if

$$(u_i, u_j) = \delta_{ij} \equiv \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Theorem 11.4.4 Let X and Y be finite dimensional inner product spaces. Then $\mathcal{L}(X, Y)$ is a vector space with addition and scalar multiplication defined naturally by:

$$(A + B)(x) \equiv Ax + Bx$$

$$(\alpha A)(x) \equiv \alpha(Ax)$$

where $A, B \in \mathcal{L}(X, Y)$ and $\alpha \in \mathbb{F}$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for X and $\{w_1, \ldots, w_m\}$ be an orthonormal basis for Y. Then a basis for $\mathcal{L}(X, Y)$ is

$$\{w_j \otimes v_i : i = 1, ..., n, j = 1, ..., m\}$$

Example 3.1

Let $\{\hat{x}, \hat{y}, \hat{z}\}$ be the standard basis in $\mathbb{R}^3$. Construct a basis for $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ in terms of the standard basis vectors.

By Theorem 11.4.4, a basis for $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ is:

$$\beta = \{\hat{x} \otimes \hat{x}, \hat{x} \otimes \hat{y}, \hat{x} \otimes \hat{z}, \hat{y} \otimes \hat{x}, \hat{y} \otimes \hat{y}, \hat{y} \otimes \hat{z}, \hat{z} \otimes \hat{x}, \hat{z} \otimes \hat{y}, \hat{z} \otimes \hat{z}\}$$

Additional Note: We all know that $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$ is related to $\mathbb{R}^{3\times3}$ (the set of all 3×3 real matrices) in a direct way. This connection becomes clear when you compute the outer products given above. For concreteness, let us compute $\hat{x} \otimes \hat{x}$:

$$\hat{x} \otimes \hat{x} = \hat{x}\hat{x}^T$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly,

$$\hat{x} \otimes \hat{y} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so on. From this, it is clear that β is a basis for $\mathbb{R}^{3\times 3}$ and thus for $\mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)$.

Example 3.2

Let $v = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$. Compute the tensor product $v \otimes w$ in terms of the basis vectors in β.

$$v \otimes w = vw^T$$

$$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & -6 \\ -1 & 0 & 2 \\ 2 & 0 & -4 \end{pmatrix}$$

By our remark in Example 3.1,

$$= 3\hat{x} \otimes \hat{x} - 6\hat{x} \otimes \hat{z} - 1\hat{y} \otimes \hat{x} + 2\hat{y} \otimes \hat{z} + 2\hat{z} \otimes \hat{x} - 4\hat{z} \otimes \hat{z}$$

3 Homework