The Cavendish Experiment: Measuring the Universal Gravitational Constant G

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Two large stationary lead balls are positioned near the ends of a lead barbell suspended by a very fine fiber in a gravitational torsion balance. The two large masses exert a gravitational torque on the barbell, which oscillates like a torsional pendulum. By measuring the oscillation period and by calculating the equilibrium angle of the barbell, the universal constant of gravitation (G) is determined. In our analysis, we determined a value for the universal gravitational constant of $G = (7.0 \pm 0.6[9\%]) \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$, which agrees with the accepted value of $G = (6.67408 \pm 0.00031) \times 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$ [1] to within uncertainty.

I. INTRODUCTION

The universal gravitational constant, denoted by G, is one of the least precisely measured physical constants. As of now, we can only report six significant digits with confidence with the accepted value for G being $(6.67408\pm0.00005)\times10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}$[1]. Although measurements with record precision of up to seven significant digits were recently performed, experimental physicists are still struggling to get better consistency between various experiments measuring G [2]. While physicists still work on measuring G in the lab today, our experimental efforts to measure the gravitational interaction between masses began over two centuries ago with the Cavendish experiment, performed by Henry Cavendish between 1797 and 1798 [3]. In this report, we discuss the results of performing a modernized version of the Cavendish experiment to obtain an experimental value of the universal gravitational constant G.

The Cavendish experiment involves a torsion pendulum with two masses, each of mass m, attached to a barbell hanging on a fiber of known torsion constant. Larger masses, each of mass M, are positioned on opposite sides of the barbell and allowed to swing from one side to the other. The top-down view of the torsion pendulum, barbell, and four masses is shown in Figure 1. Only the forces caused by one of the larger masses are shown in the figure, but we note that the magnitudes of the forces are the same for the other mass due to symmetry. By Newton's law of universal gravitation [4, p. 403],

$$F_1 = G\frac{Mm}{a^2}, \tag{1}$$

$$F_2 = G\frac{Mm}{a^2 + (2d)^2}. \tag{2}$$

Then, the corresponding torques, given by $\vec{\tau} = \vec{r} \times \vec{F}$ [4, p. 310], produced are:

$$\tau_1 = G\frac{Mmd}{a^2}, \tag{3}$$

$$\tau_2 = -G\frac{Mmad}{(a^2 + 4d^2)^{3/2}}, \tag{4}$$

taking clockwise to be the direction of positive torque. Noting that the same torques τ_1 and τ_2 are caused by the second larger mass, we see that the net torque is given by:

$$\tau = \frac{2GmMd}{a^2}\left(1 - \left[\frac{a}{\sqrt{4d^2 + a^2}}\right]^3\right). \tag{5}$$

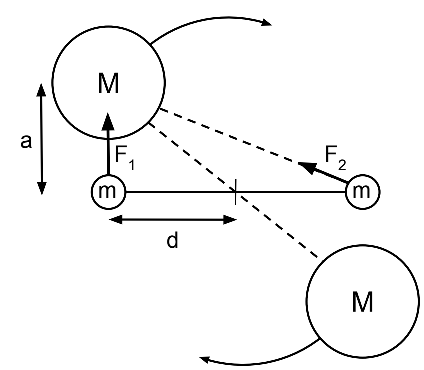

FIG. 1. This is a schematic diagram of the top-down view of the torsion pendulum with two small masses m and larger masses M set in one of the two positions.

For a torsion pendulum, the restoring torque is proportional to the angular displacement [4, p. 451]:

$$\tau = k\theta_{\mathrm{eq}}, \tag{6}$$

where τ is the magnitude of the restoring torque, k is the torsion constant of the fiber, and θ_{eq} is the angular displacement.

Thus, if the system is allowed to reach equilibrium in the orientation shown in Figure 1, the torque in Equation (5) is exactly balanced by the restoring torque in

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Equation (6):

$$k\theta_{\text{eq}} = \frac{2GmMd}{a^2}\left(1 - \left[\frac{a}{\sqrt{4d^2+a^2}}\right]^3\right). \tag{7}$$

Now, for a torsion pendulum the angular frequency of the oscillation is given by $\omega = \sqrt{\frac{k}{I}}$ [8, p. 54], where k is the torsion constant and I is the moment of inertia. Hence,

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{k}}. \tag{8}$$

Rearranging, we obtain:

$$k = \frac{4\pi^2 I}{T^2} \tag{9}$$

Substituting Equation (9) into Equation (7) and solving for G, we get:

$$G = \frac{2\pi^2 I\theta_{\text{eq}}a^2}{T^2 mMd\left(1 - \left[\frac{a}{\sqrt{4d^2+a^2}}\right]^3\right)} \tag{10}$$

Equation (10) is the main equation of our experiment, used to determine an experimental value of G based on our measurements.

II. EXPERIMENTAL SKETCH AND SALIENT DETAILS

II.1. Procedure

In addition to the torsion pendulum, our apparatus also included a laser, which was positioned in front of the torsion pendulum, and a long ruler, which we placed on a distant wall. The laser was pointed towards a small mirror which was at the center of the barbell in the torsion balance. The laser was adjusted so that it was reflected from the barbell's mirror to the ruler on the wall. Figure 2 contains a sketch of how our apparatus appeared as seen from above. At the beginning of the experiment, we positioned the torsion balance on a stable base in order to prevent external oscillations from affecting our measurements. We also made sure that the torsion balance was tilted in such a way that the barbell could rotate freely, without touching any part of the torsion balance other than the fiber that suspended it. This was crucial to avoid any additional friction which would have negatively affected our data. Once we were satisfied with the position of the torsion balance, we placed the large masses in position 1 and we left our apparatus undisturbed for a few days in order to let all the internal vibrations die off. Once the barbell had settled, we quickly rotated the large masses from position 1 to position 2. At this point, the barbell started rotating around the new equilibrium angle (θ_{eq}). As the barbell rotated, the laser light on the ruler moved. We recorded the position of the laser every ten seconds for roughly forty minutes. After the barbell had settled in position 2, we rotated the large masses back to position 1 and we measured the position of the laser by following the same procedure as before. Once we had collected enough data to obtain smooth plots of the motion of the laser, we measured the distance between the torsion balance and the ruler (L), the radius of the large mass (R_M), and the thickness of the part of the torsion balance which contained the barbell (A). We also measured the mass of the larger balls (M) with a balance scale. We weren't able to directly measure the mass of the smaller balls (m), the radius of the smaller masses (R_m), and the distance between the center of the small masses and the center of the barbell's bar (d), because we were not allowed to open the torsion balance.

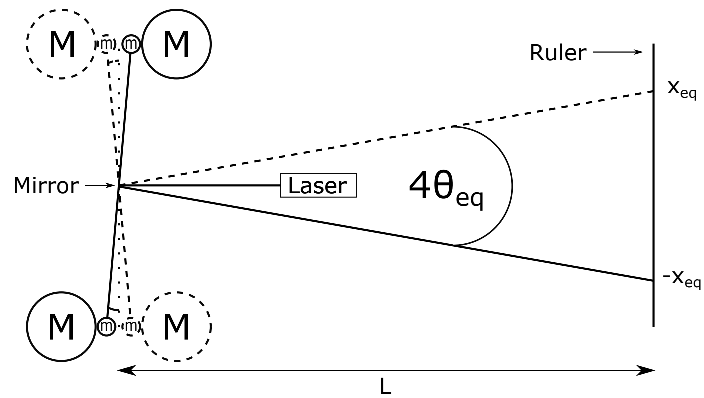

FIG. 2. Above view of our apparatus. The solid line shows the apparatus in position 1, while the dashed line shows the apparatus in position 2. When the apparatus in in position 1, the barbell is at $-\theta_{\text{eq}}$ and the laser is at $-x_{\text{eq}}$. When the apparatus in in position 2, the barbell is at θ_{eq} and the laser is at x_{eq}.

II.2. Data Analysis

Once we had collected our data, we obtained the plots of the position of the laser on the ruler versus time for both position 1 and position 2. Since the laser is reflected off a mirror positioned on a torsion pendulum, we know that the motion of the laser on the ruler should be harmonic. However, the large masses produced gravitational damping which decreased the angle of rotation of the barbell. Because of this, the amplitude of the oscillations noticeably decreased as time passed. Since the damping is light, our characterization of the position of the laser point should be of an under-damped oscillator given by

$$x = Ae^{-bt}\cos(\omega_d t + \alpha), \tag{11}$$

where A is the initial amplitude of oscillation, b is the damping constant, ω_d is the angular frequency of the damped oscillation, α is some initial phase, and x is measured with respect to the equilibrium position [8, p. 65]. For the remainder of this report, we shall denote the frequency or period of the damped oscillation with the subscript d to distinguish them from the free oscillation frequency and period.

Thus, we decided to fit our data to a damped sinusoidal curve of the form

$$f_{\text{fit}}(t) = p_3\, e^{-p_1 t} \cos\left(\frac{2\pi t}{p_2} + p_5\right) + p_4, \tag{12}$$

where the parameters p_i are our fit parameters. In this equation, p_1 is the damping constant, p_2 is the period of oscillation of the damped system, p_3 is the initial amplitude, p_4 is the offset, and p_5 is the phase. We were particularly interested in the values of p_1, p_2, and p_4 since they respectively correspond to the damping constant, the period of the oscillation, and x_{eq}, which is the x-coordinate of the laser when the barbell is at the equilibrium angular displacement θ_{eq}.

However, notice that in our theory we have assumed that the oscillations of the torsion pendulum, and hence the corresponding oscillations of the laser point, are undamped. In other words, we desire the period of the free oscillation T to compute G whereas our p_2 parameter gives us the period for the damped oscillation T_d instead. The angular frequencies of the free oscillation and the damped oscillation are related by:

$$\omega_d^2 = \omega^2 - b^2, \tag{13}$$

where ω_d is the angular frequency of the damped oscillation, ω is the angular frequency of the free oscillation, and b is the damping constant [8, p. 65]. Using the relationship between the period and the angular frequency $T = \frac{2\pi}{\omega}$, we calculate the period of the free oscillation by:

$$T = \frac{2\pi}{\sqrt{\left(\frac{2\pi}{T_d}\right)^2 + b^2}}, \tag{14}$$

where once again the damping constant b and the period of damped oscillation T_d are given by our parameters p_1 and p_2 respectively. Since we swung the pendulum twice, we obtained two similar but different values for T_d and thus T as calculated in Equation (14). When we calculated G with equation (10), we used the average of these two values of T. Similarly, when we measured the mass and the radius of the larger balls, we obtained two values for both M and R_m. When we computed equation (10) we used the average of these values. As for the equilibrium position, as indicated in Figure 2, we defined x_{eq} to be the equilibrium position of the laser when the large masses were in position 2, and we defined $-x_{\text{eq}}$ to be the position of the laser when the large masses were in position 1. We calculated the equilibrium angle θ_{eq} with the equation

$$\theta_{\text{eq}} = \frac{1}{4} \arcsin\left(\frac{\Delta x_{\text{eq}}}{\sqrt{(\Delta x_{\text{eq}})^2 + (L)^2}}\right), \tag{15}$$

where L is the distance between the torsion balance and the ruler, and Δx_{eq} is the distance between the two equilibrium positions x_{eq} and $-x_{\text{eq}}$, i.e., $\Delta x_{\text{eq}} = |x_{\text{eq}} - (-x_{\text{eq}})|$. The factor of one-fourth in equation (15) is due to two things: first, in equation (15) we used Δx_{eq}, which is twice the shift from the equilibrium position of the laser without the large masses placed on the torsion balance; second, when light is reflected off a mirror, the angle of rotation of the reflected beam is double the angle of rotation of the mirror [5, p. 183].

At this point we knew all the values of the parameters in Equation (10), except the moment of inertia of the barbell (I). Since the barbell is composed by a cylindrical bar and two balls located at the ends of the bar, according to the parallel-axis theorem [6, p. 321] the moment of inertial of the barbell is

$$I = I_{\text{bar}} + 2\left(I_{\text{ball}} + md^2\right), \tag{16}$$

where I_{bar} is the moment of inertia of the bar around an axis that passes through its center of mass, I_{ball} is the moment of inertia of a ball around an axis that passes through its center of mass, d is the distance between the center of one ball and the center of the bar, and m is the mass of one ball. Now, I_{bar} is

$$I_{\text{bar}} = \frac{1}{12}\, m_{\text{bar}}\, L_{\text{bar}}^2, \tag{17}$$

where m_{bar} and L_{bar} are respectively the mass of the bar and the length of the bar, and I_{ball} is

$$I_{\text{ball}} = \frac{2}{5}\, m_{\text{ball}}\, r_{\text{ball}}^2, \tag{18}$$

where m_{ball} and r_{ball} are respectively the mass and the radius of the ball [6, p. 318]. By inspecting these equations and considering that in our apparatus the mass of the bar was considerably smaller than the mass of the two balls, we decided to ignore I_{bar}. Thus, by substituting equation (18) into equation (16) and by using the parameters relevant to our apparatus, we obtain

$$I = \frac{4}{5} m R_m^2 + 2md^2, \tag{19}$$

where m is the mass of one ball, d is the distance between the center of one ball and the middle of the bar, and R_m is the radius of the small balls. Finally, we use Equation (10) to obtain our experimental value of the universal gravitational constant G.

II.3. Error Analysis

As previously mentioned, we were not able to directly measure the mass of the smaller balls (m), the radius of the smaller masses (R_m), and the distance between the center of the small masses and the center of the barbell's bar (d). Fortunately, the instruction manual of the torsion balance reported the values of these constants.

Judging from what we could see inside the torsion balance, the values provided are reliable. Furthermore, the instruction manual also reported the value of the mass of the large balls (M). The value reported was $M = 1.50$kg, and the value we measured is $M = (1.508 \pm 0.005)$kg. Because of this, we decided to trust the values reported in the manual and we picked relatively low uncertainties for them. Specifically, in our analysis the values of these constants used were $m = (2.0 \pm 0.1) \times 10^{-2}$kg, $R_m = (7.5 \pm 0.5) \times 10^{-3}$m, and $d = (5.0 \pm 0.1) \times 10^{-2}$m.

We propagated the errors following both the calculus approach and the functional approach described in the fourth chapter of the textbook *Measurements and their Uncertainties* [7]. In particular, we used the calculus approach in the cases in which the error was propagated through a reasonably simple equation which involved at most two parameters. For example, to calculate the error in Δx_{eq}, which is given by the equation $\Delta x_{\text{eq}} = |x_{\text{eq}} - (-x_{\text{eq}})|$, we used the equation

$$\alpha_{\Delta x_{\text{eq}}} = \sqrt{\left(\alpha_{x_{\text{eq}}}\right)^2 + \left(\alpha_{-x_{\text{eq}}}\right)^2}, \tag{20}$$

where $\alpha_{x_{\text{eq}}}$ and $\alpha_{-x_{\text{eq}}}$ are respectively the error in x_{eq} and $-x_{\text{eq}}$. We used a similar equation to calculate the error in averages, like the error in the average mass of the two large masses or the error in the average of the period. The only difference is that in these cases we multiplied the right-hand side of equation (20) by a factor of one half.

To calculate the error in the equilibrium angle, the moment of inertia of the barbell, and the universal gravitational constant, we used the functional approach. For instance, to find the error in the equilibrium angle, we had to first calculate $\alpha_{\theta_{\text{eq}}}^{\Delta x_{\text{eq}}}$ and $\alpha_{\theta_{\text{eq}}}^{L}$, where $\alpha_{\theta_{\text{eq}}}^{\Delta x_{\text{eq}}}$ is given by

$$\alpha_{\theta_{\text{eq}}}^{\Delta x_{\text{eq}}} = \theta_{\text{eq}}\left(\Delta x_{\text{eq}} \pm \alpha_{\Delta x_{\text{eq}}}, L\right) - \theta_{\text{eq}}\left(\Delta x_{\text{eq}}, L\right), \tag{21}$$

since θ_{eq} is a function of Δx_{eq} and L. In this equation $\alpha_{\Delta x_{\text{eq}}}$ is the error in Δx_{eq}, and we chose between the plus sign and the minus sign according to which one led to the larger error. The parameter $\alpha_{\theta_{\text{eq}}}^{\Delta x_{\text{eq}}}$ is given by a similar equation. Then, we calculated the error in θ_{eq} with the equation

$$\alpha_{\theta_{\text{eq}}} = \sqrt{\left(\alpha_{\theta_{\text{eq}}}^{\Delta x_{\text{eq}}}\right)^2 + \left(\alpha_{\theta_{\text{eq}}}^{L}\right)^2}. \tag{22}$$

To calculate the error in the moment of inertia and the universal gravitational constant, we used similar equations but with more parameters, since in our experiment both I and G are functions of several quantities. In addition, while calculating the error in G, we did not consider G to be a function of I, θ_{eq}, a ,T, m, M, and d, as indicated by equation (10). Rather, we substituted equation (19) into equation (10), so that we could write G as a function of R_m, θ_{eq}, a, T, M, and d. Doing this helped to reduce the error in G, since when we used equation (22) we did not have to take into account the error in I and m, which were relatively large.

III. CONCLUSIONS AND DISCUSSION

The plots of the fit of equation (12) to our data are reported in Figure 3 and Figure 4. Specifically, Figure 3 contains the data that we collected by moving the large masses from position 1 to position 2, while Figure 4 contains the data that we collected by moving the large masses from position 2 to position 1. As reported in Figure 3, we obtained a value of $T_d = 679.8896$s for the period of the oscillations around $-\theta_{\text{eq}}$, and we obtained a value of $-x_{\text{eq}} = 93.8528$cm for the equilibrium coordinate of the laser in position 1. Similarly, from Figure 4 we can infer that $T_d = 689.6437$s for the period of the oscillations around θ_{eq}, and $x_{\text{eq}} = 115.1061$cm for the equilibrium coordinate of the laser in position 2. Qualitatively, our fits appear to be reasonably good, since roughly half of our data points lie on the fit curve and most of the points that are not on the fit curve are close to it.

After performing the error analysis described above, the final values of parameters on the right hand side of Equation (10) were:

$$a = (0.0469 \pm 0.0008)\text{m},$$

$$d = (0.050 \pm 0.001)\text{m},$$

$$m = (0.020 \pm 0.001)\text{kg},$$

$$M = (1.508 \pm 0.005)\text{kg},$$

$$T = (680 \pm 10)\text{s},$$

$$\theta_{\text{eq}} = (0.0103 \pm 0.0007), \text{ and}$$

$$I = (0.000101 \pm 0.000006)\text{kg m}^2.$$

The resulting experimental value of the universal gravitational constant obtained was:

$$G = (7.0 \pm 0.6[9\%])\text{m}^3\text{kg}^{-1}\text{s}^{-2}.$$

We see that the true value of $G = 6.67408 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}$ falls within our uncertainty range, signifying that our experimental value agrees with the accepted value of G when we take into account the random errors in our experiment. Our experimental value of G only has a 5% error deviation from the true value, exhibiting a reasonable degree of accuracy.

The percentage uncertainty in our experimental value of G is approximately 9% which is quite significant, and shows that the precision of our experiment needs improvement. We can directly improve the precision of our experimental value by improving the precision of measurements we made in the experiment. One thing that

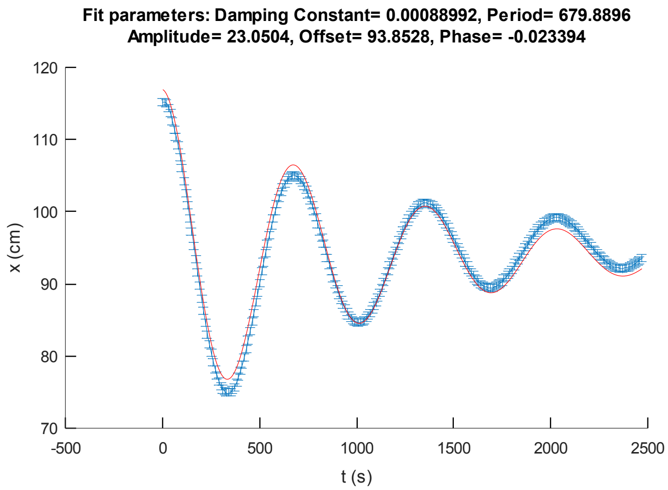

FIG. 3. Fit curve to the data that we collected by moving the large masses from position 2 to position 1. The error in the data points is $\alpha_x = 0.05$cm. The values of the fit parameters of equation (12) are reported above the plot. In this case, the value of the offset is equal to $-x_{\text{eq}}$.

contributed to our large uncertainty in our experiment is that we could not make precise measurements of quantities m, d, R_m, and a since our experimental set-up did not allow access to the barbell due to the sensitivity of our fiber. We could significantly improve the precision of our experimental value of G by measuring these quantities using precise instruments such a micro-balance and vernier caliper. We can also improve our measurement of the period by simply repeating the experimental runs and obtaining more independent measurements of the period and taking the average. When doing so, we could possibly alter the other parameters in our experiment as well. For example, we can use a different set of larger masses to change M, or change the distance between the apparatus and the wall L to further reduce our random error. Finally, the process of reading off the position of the laser point can be made more precise and with better time-resolution if we can computerize the process by using, say, a photo sensor or a camera. This further reduces the possibility of human error on top of improving the data in general. With these improvements in place, we should be able to measure the universal gravitational constant to a much better precision and accuracy.

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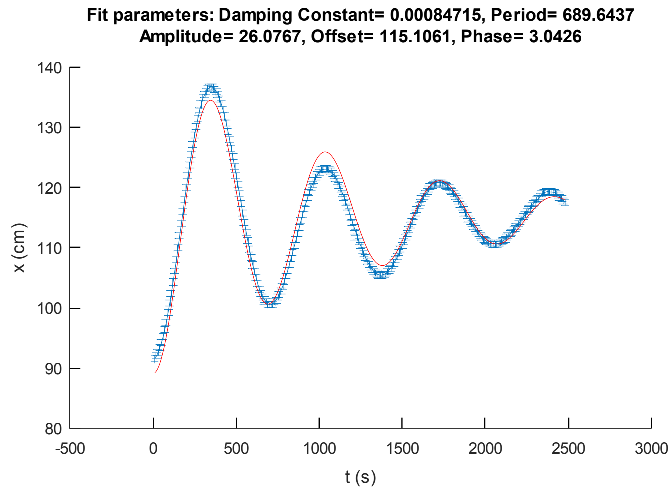

FIG. 4. Fit curve to the data that we collected by moving the large masses from position 1 to position 2. The error in the data points is $\alpha_x = 0.05$cm. The values of the fit parameters of equation (12) are reported above the plot. In this case, the value of the offset is equal to x_{eq}.