Prism Spectrometer: Speed of Light through Glass as a Function of Wavelength

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This paper reports the results of analyzing spectral lines of Helium and Mercury and the visible range using a prism spectrometer. We apply the geometry of the angle of minimum deviation to derive an equation relating the angle of minimum deviation, apex angle, the speed of light in vacuum, and the speed of light through the prism. Then, angles of minimum deviation were measured for visible spectral lines of Helium and Mercury and their wavelengths were plotted against their experimentally calculated speeds through the prism. Then, an angle of minimum deviation measurement was taken for a pair of yellow Sodium lines, and the linear fit of the plot of wavelengths versus speed of light through the prism was used to predict the wavelength of the Sodium lines and compared to the true value.

I. INTRODUCTION

When light-rays pass through a sharp boundary between two media, they slightly bend toward or away from the normal to the surface, a phenomenon known as refraction. Exactly how much the light-rays bend is governed by an equation known as Snell's law, which can be written as:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2), \tag{1}$$

where n_1 and n_2 are the refractive indices of media 1 and 2, respectively, and θ_1 and θ_2 are the angles between the normal to the surface for media 1 and 2, respectively. Recall that the speed of light in a medium is given by $v = c/n$ where c is the speed of light in vacuum and n is the refractive index of the medium. Substituting this relationship into Equation 1 yields

$$\begin{aligned} \frac{c}{v_1}\sin(\theta_1) &= \frac{c}{v_2}\sin(\theta_2) \\ \implies \frac{\sin(\theta_1)}{\sin(\theta_2)} &= \frac{v_1}{v_2}. \end{aligned} \tag{2}$$

For minimum deviation, we want the light-ray to pass through the prism symmetrically as shown in Figure 1. In other words, we would like the exit angle, the angle between the normal to the surface of the prism and the exiting light-ray, to be equal to the incident angle, θ_i. The reason we require this symmetry for minimum deviation is because we know that the minimum deviation angle is unique and that laws of optics are time reversible. If the minimum deviation occurred when the light-ray was not passing through the prism symmetrically, we would be able to reverse time and show that there exists a second minimum deviation angle, which is not true.

From the first boundary, going from vacuum into the prism, we can apply Equation 2 to write:

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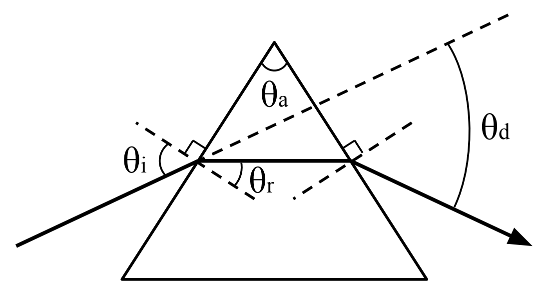

FIG. 1. This is a schematic representation of a light-ray passing through a prism symetrically, where θ_a is the apex angle, θ_d is the deviation angle, θ_i is the incident angle, and θ_r is the angle of refraction.

$$\begin{aligned} \frac{\sin(\theta_i)}{\sin(\theta_r)} &= \frac{c}{v} \\ \implies v &= c \times \frac{\sin(\theta_r)}{\sin(\theta_i)}. \end{aligned} \tag{3}$$

where v is the speed of light in prism. Exploiting the symmetry of the situation, we notice that the light-ray passing through the prism creates an isosceles triangle with the two surfaces of the prism adjacent to the apex angle. This allows us to say that the angle between one of the surfaces and the ray inside the prism is exactly $(90^\circ - \frac{1}{2}\theta_a)$. Since this angle and θ_r must sum to a right angle, this implies that

$$\theta_r = \frac{1}{2}\theta_a. \tag{4}$$

The angle of deviation, θ_d, is the angle between the exiting ray and the incident ray, which can be expressed in terms of θ_i and θ_r. At the first boundary, the light-ray changes its direction by an angle of $(\theta_i - \theta_r)$. Once again, by applying symmetry, we notice that the incident angle at the second boundary is equal to θ_r and that the exiting angle is equal to θ_i. Thus, the light-ray changes its direction by exactly $(\theta_i - \theta_r)$ again at the second boundary. Hence, we may write

$$\theta_d = 2(\theta_i - \theta_r).$$

Substituting Equation 4 in and rearranging for θ_i, we obtain

$$\theta_i = \frac{1}{2}[\theta_a + \theta_d] \tag{5}$$

By substituting Equations 4 and 5 into Equation 3, we get a useful result:

$$v = c \times \frac{\sin(\frac{1}{2}\theta_a)}{\sin(\frac{1}{2}[\theta_a + \theta_d])}. \tag{6}$$

In this lab, we study the spectra of Helium, Mercury, and Sodium by using a prism spectrometer, a device that can accurately measure the deviation angle of particular wavelengths of light.

II. PROCEDURE

II.1. Initial Adjustments

A horizontal line was marked on the whiteboard for adjustment of the telescope and the collimator to be on the same horizontal plane using a second, external telescope. No adjustment of screws under the telescope was necessary for this horizontal alignment. Then, the focus on crosshair position of the telescope was adjusted to the collimator using a standard LED light bulb as our light source. Once these adjustments were complete, none of the screws that alter the horizontal alignment, the focus of the telescope, or the vertical alignment of the crosshair were touched throughout the remainder of the experiment.

II.2. Finding the Apex Angle

Once our choice of apex angle is labeled, we measure it precisely by pointing it toward an illuminated slit and measuring the angle of the two reflected rays as shown in Figure 2 using our prism spectrometer. As in Figure 2, bisect the apex angle by a line segment parallel to our incident rays, and call the angles a and b. Since the bisecting line segment is parallel to the incident rays by assumption, the angles between the surfaces of the prism adjacent to the apex angle and the incident rays are a and b also. Then, by the law of reflection, the reflected rays will also make angles a and b with the two surfaces. From this, we immediately see that we can write the angle between the two reflected rays as:

$$\begin{aligned}\theta &= a + a + b + b \\ &= 2(a+b) \\ &= 2\theta_a.\end{aligned} \tag{7}$$

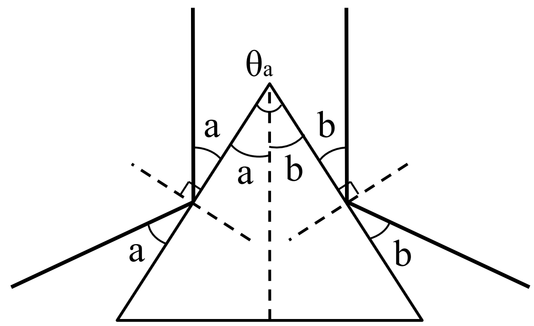

FIG. 2. This is a schematic representation of the experimental set-up to find the apex angle.

Therefore, measuring the angle between the reflected rays using the prism spectrometer allows us to calculate the apex angle to a high degree of precision.

II.3. Measurements for Helium, Mercury, and Sodium

For each of Helium, Mercury, and Sodium, the movement of the positions of the spectral lines with small rotations of the prism was observed to ensure that we measure the angle of minimum deviation for each spectral line. For Helium and Mercury, known spectra were used to aid the process of identifying spectral lines, and once identified, the angle of minimum deviation was measured for each spectral line using the vernier scale of the prism spectrometer. Using Equation 6, the speed of that particular wavelength of light through the prism was calculated for each spectral line, and the speed of light through the prism was plotted against wavelength. Finally, we measure the angle of minimum deviation of the yellow Sodium lines and use our graph of speed of light against wavelength to predict the wavelength of the Sodium lines to test for the experiment's accuracy.

III. RESULTS

III.1. Initial Measurements

Using the standard LED light bulb as our light source, we found that the two reflected rays were at angles $(49.783 \pm 0.008)^\circ$ and $-(52.6 \pm 0.3)^\circ$, where the significantly larger uncertainty in our second measurement is due to significant width of the second reflected ray. From this, our apex angle was calculated to be:

$$\theta_a = \frac{49.783^\circ - (-52.6^\circ)}{2} = (51.2 \pm 0.3)^\circ.$$

Since all angles of deviation are measured with respect to the position of the collimator, we must take a measurement of the zero-angle, and adjust all of our angle measurements according to the zero-angle. The zero-angle was measured to be:

$$\theta_0 = -(0.850 \pm 0.008)^\circ.$$

III.2. Data for Helium

Angles of minimum deviation measurements for each of the visible spectral lines of Helium, identified by the color and wavelength data provided by NIST are shown below in a table.

Color	Wavelength, nm	Angle of minimum deviation, deg
dark violet	402.6	51.68
light violet	438.8	49.95
dark blue	447.1	49.93
light blue	471.3	49.50
dark green	492.2	49.13
light green	501.6	49.00
yellow	587.6	48.02
light red	667.8	47.45
dark red	728.1	47.28

TABLE I. Color, Wavelength, and Angle of Minimum Deviation Measurements for Visible Spectral Lines of Helium

The uncertainty on the angles of minimum deviation measurements are approximately $\pm 0.02^\circ$.

III.3. Data for Mercury

The same data for visible spectral lines of Mercury are displayed in a table below.

Once again, the uncertainty on the angles of minimum deviation measurements are around $\pm 0.02^\circ$.

Color	Wavelength, nm	Angle of minimum deviation, deg
purple	407.8	51.03
blue	435.8	50.20
dark green	491.6	49.13
light green	546.1	48.45
yellow	577.0	48.18
red	623.4	47.45

TABLE II. Color, Wavelength, and Angle of Minimum Deviation Measurements for Visible Spectral Lines of Mercury

III.4. Analysis of Data for Helium and Mercury

Now, using our angles of minimum deviation measurements for visible spectral lines of Helium and Mercury, we calculate the speed of each particular wavelength through the prism using Equation 6:

$$v = c \times \frac{\sin(\frac{1}{2}\theta_a)}{\sin(\frac{1}{2}[\theta_a + \theta_d])}.$$

All the wavelengths of visible spectral lines of Helium and Mercury and their experimental speed of light through the prism are shown in the table below.

Speed of Light through the Prism, $\times 10^7$ m/s	Wavelength, nm
16.57	402.6
16.64	407.8
16.74	435.8
16.77	438.8
16.77	447.1
16.83	471.3
16.87	491.6
16.87	492.2
16.89	501.6
16.96	546.1
16.99	577.0
17.01	587.6
17.08	623.4
17.08	667.8
17.10	728.1

TABLE III. Speed of Light through the Prism for Wavelengths of Visible Spectral Lines of Helium and Mercury

The uncertainties calculated for the experimental values of speed of light through the prism are quite large, coming out to about $\pm 3 \times 10^7$ m/s. In Table III, we have reported more decimal places for distinguishability, and since we believe our results are still reasonably precise although not necessarily accurate. The reason is because a significant contribution of our error is coming from our large uncertainty in the measurement of the apex angle, which is common to all wavelengths. A large deviation of our measurement of the apex angle from its true value would shift each of our calculated speeds in much the same way, affecting our accuracy but not our precision.

The wavelength was plotted against the speed of light through the prism, and the plot is shown in Figure 3. No error bars are shown, since the vertical error bars are negligible (using NIST reported values of 10^{-13}m or 10^{-14}m precision), and the horizontal error bars are significantly larger than the horizontal scale displayed, due to our large uncertainty in the apex angle once again. Despite the large uncertainty in the experimental speed of light values, we see a linear trend, supporting our theory that our set of data is still reasonably precise.

By linear regression, we get the following slope and intercept for our line of best fit:

$$m = (5.7 \pm 0.6) \times 10^{-5}\ \frac{nm \cdot s}{m}$$

$$c = -(9.1 \pm 0.9) \times 10^{3}\ nm$$

III.5. Predicting the Wavelength of the Sodium Yellow Lines

The same procedure was applied to make a measurement of the angle of minimum deviation of a pair of yellow Sodium spectral lines. The angle of minimum deviation measured was:

$$(48.03 \pm 0.02)^\circ.$$

According to this data, we once again calculate the speed of light of this particular wavelength through the prism using Equation 6, yielding:

$$(1.7 \pm 0.3) \times 10^{8} m/s.$$

According to Goggin, the wavelengths of the two yellow Sodium lines are (589.2 ± 0.1)nm and (589.8 ± 0.1)nm, but since we cannot resolve the two separately, let us take them to be one line at the average of the two lines with the corresponding larger uncertainty: (589.5 ± 0.3)nm. We will use our plot of wavelength against speed of light through the prism to predict the wavelength of this pair of yellow Sodium lines and see how accurate our experiment was. Using our line of best fit,

$$\begin{aligned}\lambda &= 5.7 \times 10^{-5} nm \cdot s/m \times 1.7 \times 10^{8} m/s - 9.1 \times 10^{3} nm \\ &= 595 nm.\end{aligned}$$

Methods of linear regression would predict an uncertainty of this wavelength greater than the value itself, rendering this prediction meaningless, but once again, this uncertainty is blown up out of proportion due to our large uncertainty in the measurement of the apex angle. Since our results have generally contained uncertainties of about $\pm 10\%$, we simply assume that to be our uncertainty of our prediction of the wavelength here, and report our final result to be:

$$\lambda = (600 \pm 60) nm.$$

We immediately see that the true value of the wavelength of the yellow Sodium lines, (589.5 ± 0.3)nm, is within our uncertainty range, suggesting that our inaccuracy is reasonably explained by random error rather than systematic error. Nonetheless, we compute our percentage error for completeness.

$$\begin{aligned}\%_e &= \frac{\lambda_{\text{exp}} - \lambda_{\text{the}}}{\lambda_{\text{the}}} \times 100\% \\ &= \frac{600 nm - 589.5 nm}{589.5 nm} \times 100\% \\ &= 1.78\%.\end{aligned}$$

This low percentage error shows that our experiment, despite large reported uncertainties, was reasonably accurate.

IV. ERROR ANALYSIS

IV.1. Uncertainty Propagation on v

Most uncertainty propagation during the data analysis were standard addition of uncertainties in quadrature. Although, one step in particular in the uncertainty propagation need justified. That is the uncertainty propagation through Equation 6:

$$v = c \times \frac{\sin(\frac{1}{2}\theta_a)}{\sin(\frac{1}{2}[\theta_a + \theta_d])}.$$

Here, we use the calculus approximation:

$$(\alpha_v)^2 = \left(\frac{\partial v}{\partial \theta_a}\right)^2 (\alpha_{\theta_a})^2 + \left(\frac{\partial v}{\partial \theta_d}\right)^2 (\alpha_{\theta_d})^2, \qquad (8)$$

where we have considered the uncertainty of the speed of light in vacuum negligible. Note that

$$\begin{aligned}\frac{\partial v}{\partial \theta_a} &= c \times \frac{\frac{1}{2}\cos(\frac{1}{2}\theta_a)\sin[\frac{1}{2}(\theta_a + \theta_d)] - \frac{1}{2}\cos[\frac{1}{2}(\theta_a + \theta_d)]\sin(\frac{1}{2}\theta_a)}{\sin^2[\frac{1}{2}(\theta_a + \theta_d)]} \\ &= \frac{1}{2} c \frac{\sin(\frac{1}{2}\theta_d)}{\sin^2[\frac{1}{2}(\theta_a + \theta_d)]},\end{aligned}$$

and similarly,

$$\frac{\partial v}{\partial \theta_d} = -\frac{1}{2} c \frac{\sin(\frac{1}{2}\theta_a)\cos[\frac{1}{2}(\theta_a + \theta_d)]}{\sin^2[\frac{1}{2}(\theta_a + \theta_d)]}.$$

We have determined the error propagation on v by substituting these expressions in Equation 8.

IV.2. Linear Regression

From the data points of the wavelength against speed of light through the prism graph, we determined the uncertainties of the slope and intercept of our line of best

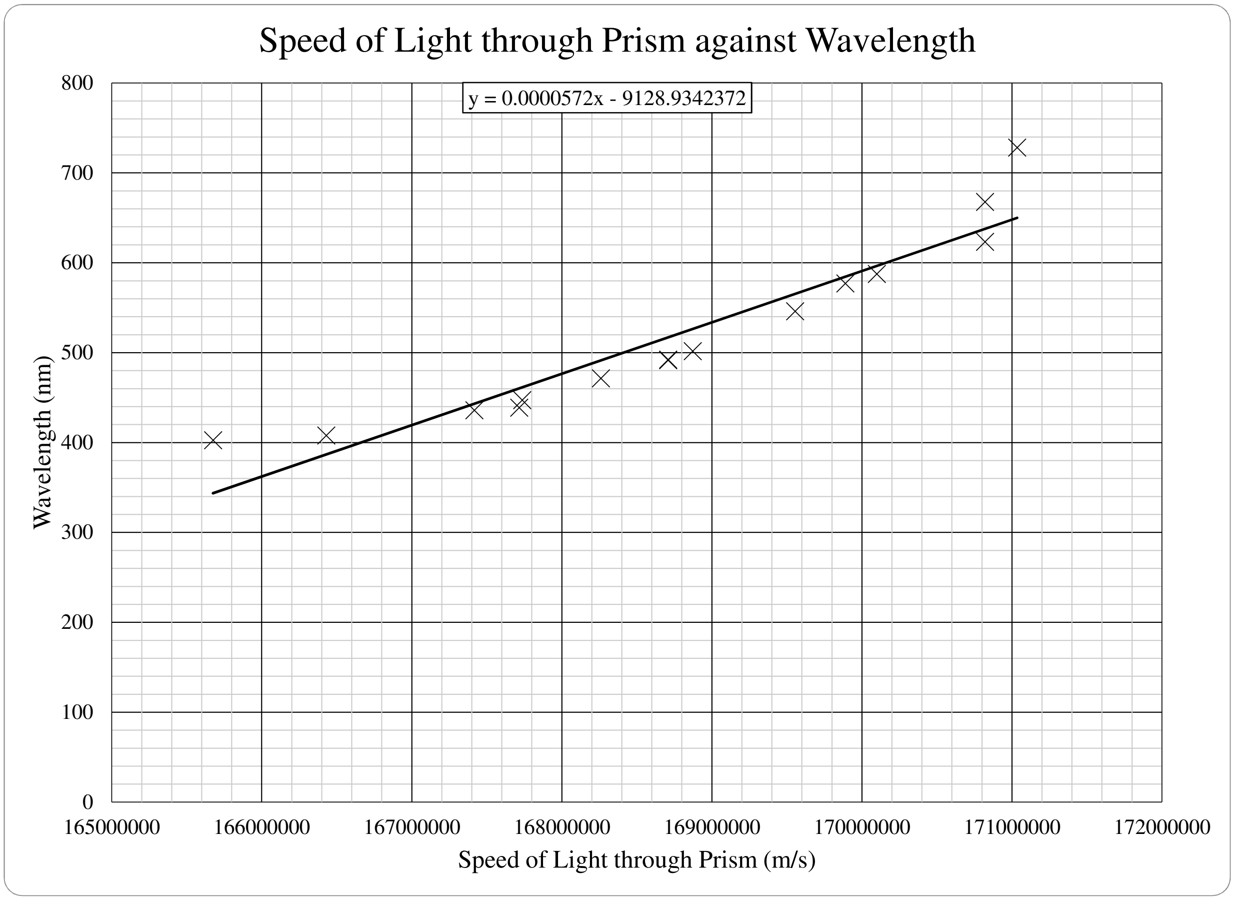

FIG. 3. This is a plot of wavelength against the speed of light through the prism for visible spectral lines of Helium and Mercury.

fit using the equations:

$$\alpha_c = \alpha_{CU}\sqrt{\frac{\sum_i v_i^2}{\Delta}},$$

and

$$\alpha_m = \alpha_{CU}\sqrt{\frac{N}{\Delta}},$$

where

$$\Delta = N\sum_i v_i^2 - \left(\sum_i v_i\right)^2,$$

and

$$\alpha_{CU} = \sqrt{\frac{1}{N-2}\sum_i(\lambda_i - mv_i - c)^2},$$

following the methods of Hughes & Hase.

V. CONCLUSION

In this experiment, we successfully measured visible spectral lines of Helium and Mercury, and produced a graph of wavelength versus speed of light through the prism of the particular wavelengths seen as the spectral lines of the two gases, and furthermore predicted the wavelength of yellow Sodium lines using our graph to a reasonable degree of accuracy. We first derived a relationship between the angle of minimum deviation, θ_d, the apex angle, θ_a, the speed of light in vacuum, c, and the speed of light in the prism, v by using the geometry of the experimental set-up. After careful adjustments to make sure we are measuring the angle of minimum deviation, sets of data were taken for both Helium and Mercury spectral lines. Using the derived equation, we calculated the experimental speed through the prism for each wavelength of light observed, and plotted the wavelength against the speed of light through the prism. The equation of the line of best fit was calculated to be: $\lambda = 5.7 \times 10^{-5} v - 9.1 \times 10^{3}$. Using this relationship, we calculated an experimental prediction of the wavelength of the yellow Sodium lines from a measurement of its angle of minimum deviation. The experimental prediction of the wavelength was (600 ± 60)nm where the true value was (589.5 ± 0.3)nm. The true value falls within our uncertainty region, showing no sign of systematic error, and our percentage error calculates to 1.78% indicating that our experimental results are of reasonably accuracy.

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