Permeability of Free Space (μ_0) and the Force between Two Current-Carrying Wires (Current Balance)

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(Dated: January 31, 2019)

This paper reports the results of three data sets of an experiment using the current balance to measure the permeability of free space (μ_0). We apply Ampere's Law to derive the formula $F = \frac{\mu_0 L I^2}{2\pi d}$ for the magnetic force between two wires carrying equal currents in opposing directions. Then, we fix d and L, and measure the current required to balance the weight of small, known masses. As we vary the masses, we obtain a linear relationship between F and I^2. Using the slope of the graph plotting these two quantities, we compute an estimate for the constant $\mu_0/2\pi$.

I. INTRODUCTION

Parallel current-carrying wires exert magnetic forces on each other. This is because the magnetic field produced by one of the currents interacts with the other current and vice versa. By applying our two right-hand rules to determine the direction of the magnetic field generated and the direction of the magnetic force, we notice that the wires attract one another if the currents run in the same direction, and repel each other if the currents run in opposite directions, as show in Figure 1.

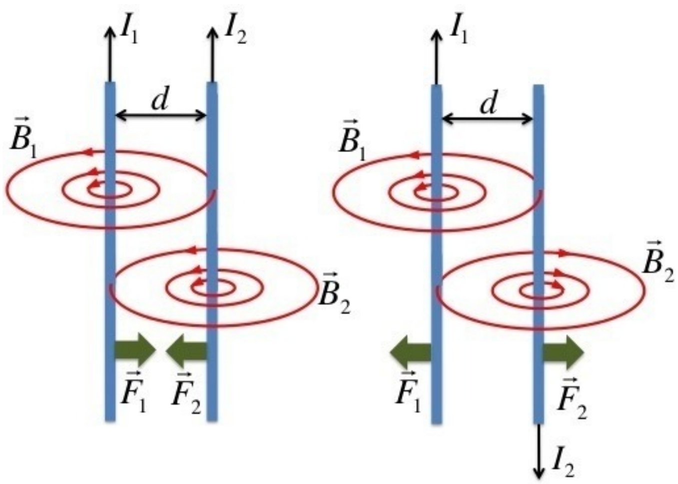

FIG. 1. This is a schematic representation of two parallel current-carrying wires, the magnetic fields generated, and the magnetic forces experienced by the wires as result.

Now, to determine the magnitude of the magnetic field generated by a current, we apply Ampere's Law under the assumption that the length of the wire, L, is significantly larger than the distance between the two wires, d ($L >> d$), so that we can treat the wire as an infinitely long line of charge. Ampere's Law in the case of a static electric field, in integral form, can be written as:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I, \tag{1}$$

stating that the line integral of the magnetic field over a closed loop is equal to the current enclosed multiplied by the permeability of free space, the fundamental physical constant of our interest. Fortunately for us, the geometry of the system makes the line integral easily computable. As shown in Figure 1, the magnetic field produced by a straight wire forms concentric rings, so if we are interested in the magnetic field strength a distance d away from the wire, we are looking at a circular loop of radius d. Furthermore, we notice that an infinitesimal line element $d\vec{s}$ is always parallel to the magnetic field $\vec{B}$ at any point along the loop and that the magnetic field strength must be constant along the loop by symmetry. The result is that the left-hand side of our Equation 1 simply turns into a product of the magnetic field strength and the circumference of our loop, giving us:

$$\begin{aligned} B \times 2\pi d &= \mu_0 I \\ \implies B &= \frac{\mu_0 I}{2\pi d}. \end{aligned} \tag{2}$$

Then, since this is the magnetic field strength at the location of the other current, we can calculate the magnetic force experienced by the other current by our basic magnetic force equation:

$$\vec{F} = \vec{I} L \times \vec{B}. \tag{3}$$

Since we have already determined the direction of the force via the right-hand rule, and we see that the magnetic field is perpendicular to the current, we may simply write the magnitude of the magnetic force as:

$$\begin{aligned} F &= ILB \\ \implies F &= IL\left(\frac{\mu_0 I}{2\pi d}\right) \\ \implies F &= \frac{\mu_0 L I^2}{2\pi d}. \end{aligned} \tag{4}$$

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This final equation is central to our experiment, as we will be balancing this repelling magnetic force between two forces with the weight of small masses. For one set of measurements, we fix d and L, and measure the current required to balance the weight as we vary the small masses.

II. PROCEDURE

The current balance is an experimental device that allows us to send currents through two parallel wires. The bottom wire is fixed and the top wire, which sits directly above the bottom wire, is movable with a small tray for masses to be placed. The two wires are checked to be parallel and completely horizontal by using a level. Equal currents are sent through the two wires in opposing directions by connecting them in series with a DC power supply capable of high currents up to $12A$. We also connect an ammeter in series to monitor the current through the circuit. Now, in order to measure the small gap between the two wires, we employ the setup represented in Figure 2. A ruler and a telescope are clamped a long distance b away from the current balance, aligned with the mirror attached to the current balance such that ruler readings can be obtained as the telescope is pointed to the mirror. Then, as the wires swing apart by a distance d_0 corresponding to a small angle θ, the mirror also swings by the same angle θ. By the law of reflection, as the vector normal to the mirror changes its direction by θ meanwhile the outgoing light ray resolved by the telescope remains in the same direction, the ingoing ray must now have changed by an angle 2θ. If we let the difference between the ruler readings when the bars are touching and the when the bars are a distance d_0 apart as y and the distance between the bars and the pivot as a as in Figure 2, we can use the geometry of this setup to obtain a relationship between y and d_0 such that we can measure d_0 more precisely.

Using trigonometry and the small-angle approximation, we see that:

$$\begin{aligned} 2\theta &\approx \frac{y}{b} \\ \theta &\approx \frac{d_0}{a} \\ \implies 2\left(\frac{d_0}{a}\right) &\approx \frac{y}{b} \\ \implies d_0 &\approx \frac{ay}{2b}. \end{aligned} \tag{5}$$

Finally, since this allows us to calculate d_0, we measure the diameters of the wires, D_u and D_d, using a micrometer and add the radius of each wire to obtain the distance between the centers of the wires. In other words,

$$d = d_0 + r_u + r_d, \tag{6}$$

where r_u and r_d are the radii of the two wires. For a set of measurements, we fix d_0, and small masses ranging from $10mg$ to $160mg$, precisely measured using the microbalance, are put on the tray atop the movable wire. For each mass, the current in the circuit is turned up until the same gap width d_0 is obtained.

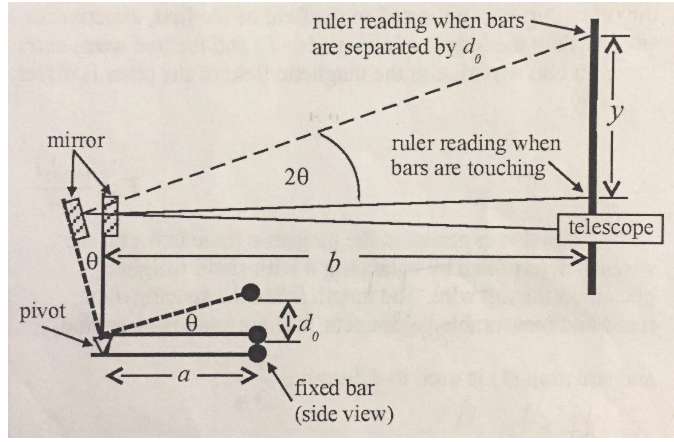

FIG. 2. This is a schematic representation of our experimental set-up. In effect, the use of the mirror and telescope allows us to amplify the small distance d_0 onto a larger ruler reading y in order to make more precise measurements of the gap between the two wires.

In order to deal with stray magnetic fields that may skew our results, we run the current in both directions and take the geometric mean of the two currents. Let us show that this is the proper way to take stray fields into account. Let the current in one direction be I_+ and the other I_-, and the stray magnetic field strength be ϵ along the direction of the magnetic field on the upper wire when the current is flowing in the I_+ direction. So, in the first direction of the current the total magnetic field strength at the top wire is given by:

$$B_+ = \frac{\mu_0 I_+}{2\pi d} + \epsilon.$$

In the other direction, stray fields will have the opposite effect on the magnetic field strength, so we introduce a negative sign:

$$B_- = \frac{\mu_0 I_-}{2\pi d} - \epsilon.$$

Now, we can calculate the magnetic force in each case:

$$F = I_+ L \left(\frac{\mu_0 I_+}{2\pi d} + \epsilon\right) \tag{7}$$

$$F = I_- L \left(\frac{\mu_0 I_-}{2\pi d} - \epsilon\right) \tag{8}$$

However, these magnetic forces are equal since they are being balanced against the same weight mg. Rearranging Equation 7, we obtain:

$$\epsilon = \frac{F}{I_+ L} - \frac{\mu_0 I_+}{2\pi d}$$

Then, substituting this expression into Equation 8, we get:

$$\begin{aligned}F &= I_-L\left(\frac{\mu_0 I_-}{2\pi d} - \left(\frac{F}{I_+L} - \frac{\mu_0 I_+}{2\pi d}\right)\right)\\ &= I_-L\left(\frac{\mu_0 I_-}{2\pi d} - \frac{F}{I_+L} + \frac{\mu_0 I_+}{2\pi d}\right)\\ &= \frac{\mu_0 L I_-}{2\pi d}(I_- + I_+) - \frac{FI_-}{I_+}.\end{aligned}$$

More rearranging,

$$\begin{aligned}\frac{F(I_- + I_+)}{I_+} &= \frac{\mu_0 L I_-}{2\pi d}(I_- + I_+)\\ F &= \frac{\mu_0 L I_+ I_-}{2\pi d}.\end{aligned} \tag{9}$$

Therefore, we shall replace I^2 in Equation 4 by the product of the two currents I_+ and I_- to remove the effects of stray fields.

III. RESULTS

III.1. Initial Measurements

Our measurements of fixed quantities in our experiment are as follows:
The length of the wires: $L = (0.2760 \pm 0.0005)m$
The distance between the wires and the pivot: $a = (0.2110 \pm 0.0005)m$
The distance between the pivot and the telescope: $b = (1.2095 \pm 0.0005)m$
The diameter of the top bar: $D_u = (0.0032100 \pm 0.0000005)m$
The diameter of the bottom bar: $D_d = (0.0032600 \pm 0.0000005)m$

III.2. Data Set 1

For our first data set, the initial ruler reading when the bars were touching was $(0.0570 \pm 0.0005)m$ and the ruler reading when the bars were apart was $(0.0650 \pm 0.0005)m$. From this, we can take the difference to yield $y = (0.008 \pm 0.001)m$. Now, using this measurement with our initial measurements, we can calculate d_0 and d by applying Equation 5 and 6.

$$\begin{aligned}d_0 &= \frac{ay}{2b}\\ &= \frac{0.2110m \times 0.008m}{2 \times 1.2095m}\\ &= (7.0 \pm 0.9) \times 10^{-4}m.\end{aligned}$$

$$\begin{aligned}d &= d_0 + r_u + r_d\\ &= 7.0 \times 10^{-4}m + \frac{0.0032100}{2}m + \frac{0.0032600}{2}m\\ &= (3.93 \pm 0.09) \times 10^{-3}m.\end{aligned}$$

The raw data for the first data set is shown below.

Mass (mg)	Forward Current, I_+ (A)	Reverse Current, I_- (A)
10.0	3.80	2.39
20.0	4.70	3.69
30.4	5.37	4.46
40.0	5.97	5.43
50.0	6.23	5.86
60.0	7.21	6.14
70.0	7.44	6.92
80.0	8.21	7.00
89.7	8.85	7.86
100.1	9.34	8.49
110.1	9.76	8.82
120.1	10.15	9.22
130.1	10.56	9.60
139.8	10.83	9.82
150.1	11.30	10.35
160.1	11.80	10.83

TABLE I. The first data set of forward and reverse currents necessary to return the ruler reading to $0.0650m$ for varying masses.

The uncertainty in the mass readings is $0.1mg$ and the uncertainty in the current readings is $0.01A$. According to these data, we plot the force of gravity (and thus the magnetic force) $F = mg$ against I_+I_- and obtain the following graph.

Using the fact that Equation 9 is in the form $y = mx+c$ if we take $y = F$, $x = I_+I_-$, $m = \frac{\mu_0 L}{2\pi d}$, and $c = 0$, we can read off the plot's slope to calculate $\mu_0/2\pi$. By methods of linear regression, the slope of the line of best-fit was calculated to be:

$$m = (1.25 \pm 0.02) \times 10^{-5} N/A^2.$$

Rearranging our expression for the slope, we obtain:

$$\begin{aligned}\frac{\mu_0}{2\pi} &= \frac{md}{L}\\ &= \frac{1.25 \times 10^{-5}N/A^2 \times 3.93 \times 10^{-3}m}{0.2760m}\\ &= (1.78 \pm 0.07) \times 10^{-7}N/A^2.\end{aligned}$$

Finally, comparing this to the actual value of $\mu_0/2\pi =$

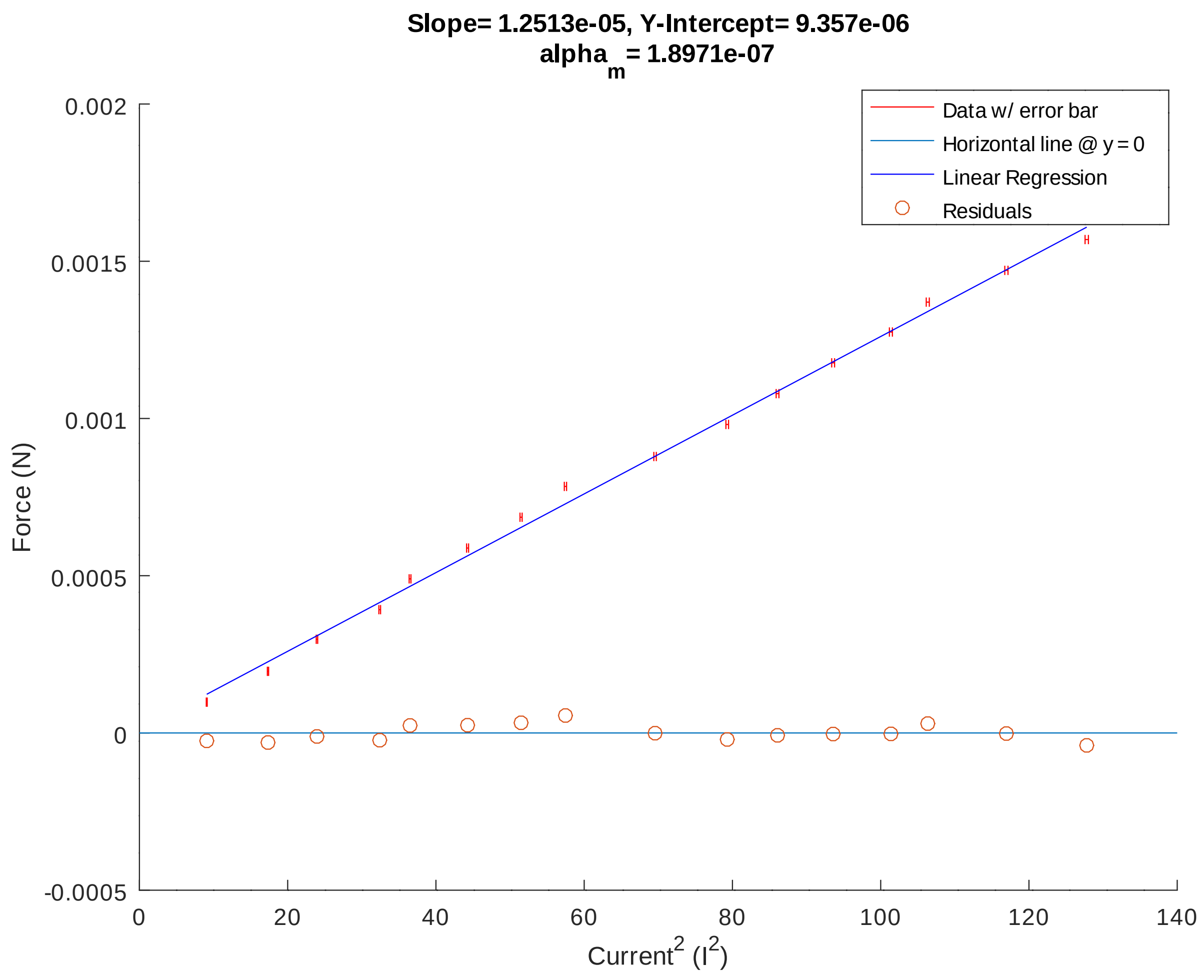

FIG. 3. This is a plot of the magnetic force (F) against the square of the current with appropriate stray-field corrections (I_+I_-). The residuals are also plotted near the horizontal axis.

$2 \times 10^{-7} N/A^2$, we obtain our percentage error.

$$\begin{aligned}\%_e &= \frac{X_{\text{exp}} - X_{\text{the}}}{X_{\text{the}}} \times 100\% \\ &= \frac{1.78 \times 10^{-7} N/A^2 - 2 \times 10^{-7} N/A^2}{2 \times 10^{-7} N/A^2} \times 100\% \\ &= 10.85\%.\end{aligned}$$

As an additional note, we qualitatively see no patterns in the residuals, which means that the $\mu_0/2\pi$ estimates do not show a statistical change as we increase the masses being placed.

III.3. Data Set 2

We apply the same analysis. For our second data set, the initial ruler reading was $(0.0590 \pm 0.0005)m$ and the ruler reading when the bars were apart was $(0.0700 \pm 0.0005)m$. From this, we can take the difference to yield $y = (0.011 \pm 0.001)m$. Now, using this measurement with our initial measurements, we can calculate d_0 and d.

$$\begin{aligned}d_0 &= \frac{ay}{2b} \\ &= \frac{0.2110m \times 0.011m}{2 \times 1.2095m} \\ &= (9.6 \pm 0.9) \times 10^{-4} m.\end{aligned}$$

$$\begin{aligned}d &= d_0 + r_u + r_d \\ &= 9.6 \times 10^{-4} m + \frac{0.0032100}{2} m + \frac{0.0032600}{2} m \\ &= (4.19 \pm 0.09) \times 10^{-3} m.\end{aligned}$$

The raw data for the second data set is shown below.

According to these data, we plot the force of gravity (and thus the magnetic force) $F = mg$ against $I_+ I_-$ and obtain the following graph.

By linear regression, the slope of the line of best-fit was calculated to be:

$$m = (1.21 \pm 0.03) \times 10^{-5} N/A^2.$$

Rearranging our expression for the slope, we obtain:

$$\begin{aligned}\frac{\mu_0}{2\pi} &= \frac{md}{L} \\ &= \frac{1.21 \times 10^{-5} N/A^2 \times 4.19 \times 10^{-3} m}{0.2760m} \\ &= (1.84 \pm 0.09) \times 10^{-7} N/A^2.\end{aligned}$$

Finally, comparing this to the actual value of $\mu_0/2\pi = 2 \times 10^{-7} N/A^2$, we obtain our percentage error.

$$\begin{aligned}\%_e &= \frac{X_{\text{exp}} - X_{\text{the}}}{X_{\text{the}}} \times 100\% \\ &= \frac{1.84 \times 10^{-7} N/A^2 - 2 \times 10^{-7} N/A^2}{2 \times 10^{-7} N/A^2} \times 100\% \\ &= 8.06\%.\end{aligned}$$

Mass (mg)	Forward Current, I_+ (A)	Reverse Current, I_- (A)
10.1	2.88	2.49
19.9	4.22	3.54
30.0	5.01	4.09
40.0	6.16	5.25
50.0	6.27	5.58
60.1	6.90	6.02
69.9	7.48	6.68
80.0	8.08	7.31
90.0	8.55	7.79
100.0	9.23	8.22
110.1	9.69	8.90
119.9	10.12	9.23
130.0	10.56	9.54
140.0	10.94	10.01
150.0	11.50	10.52
160.1	11.84	10.83

TABLE II. The second data set of forward and reverse currents necessary to return the ruler reading to $0.0700m$ for varying masses.

Once again, we qualitatively see no patterns in the residuals, which means that the $\mu_0/2\pi$ estimates do not show a statistical change as we increase the masses being placed.

III.4. Data Set 3

We once again apply the same analysis. For our third data set, the initial ruler reading was $(0.0600 \pm 0.0005)m$ and the ruler reading when the bars were apart was $(0.0815 \pm 0.0005)m$. From this, we can take the difference to yield $y = (0.022 \pm 0.001)m$. Now, using this measurement with our initial measurements, we can calculate d_0 and d.

$$\begin{aligned}d_0 &= \frac{ay}{2b} \\ &= \frac{0.2110m \times 0.022m}{2 \times 1.2095m} \\ &= (1.92 \pm 0.09) \times 10^{-3} m.\end{aligned}$$

$$\begin{aligned}d &= d_0 + r_u + r_d \\ &= 1.92 \times 10^{-3} m + \frac{0.0032100}{2} m + \frac{0.0032600}{2} m \\ &= (5.15 \pm 0.09) \times 10^{-3} m.\end{aligned}$$

The raw data for the third data set is shown below.

According to these data, we plot the force of gravity (and thus the magnetic force) $F = mg$ against $I_+ I_-$ and obtain the following graph.

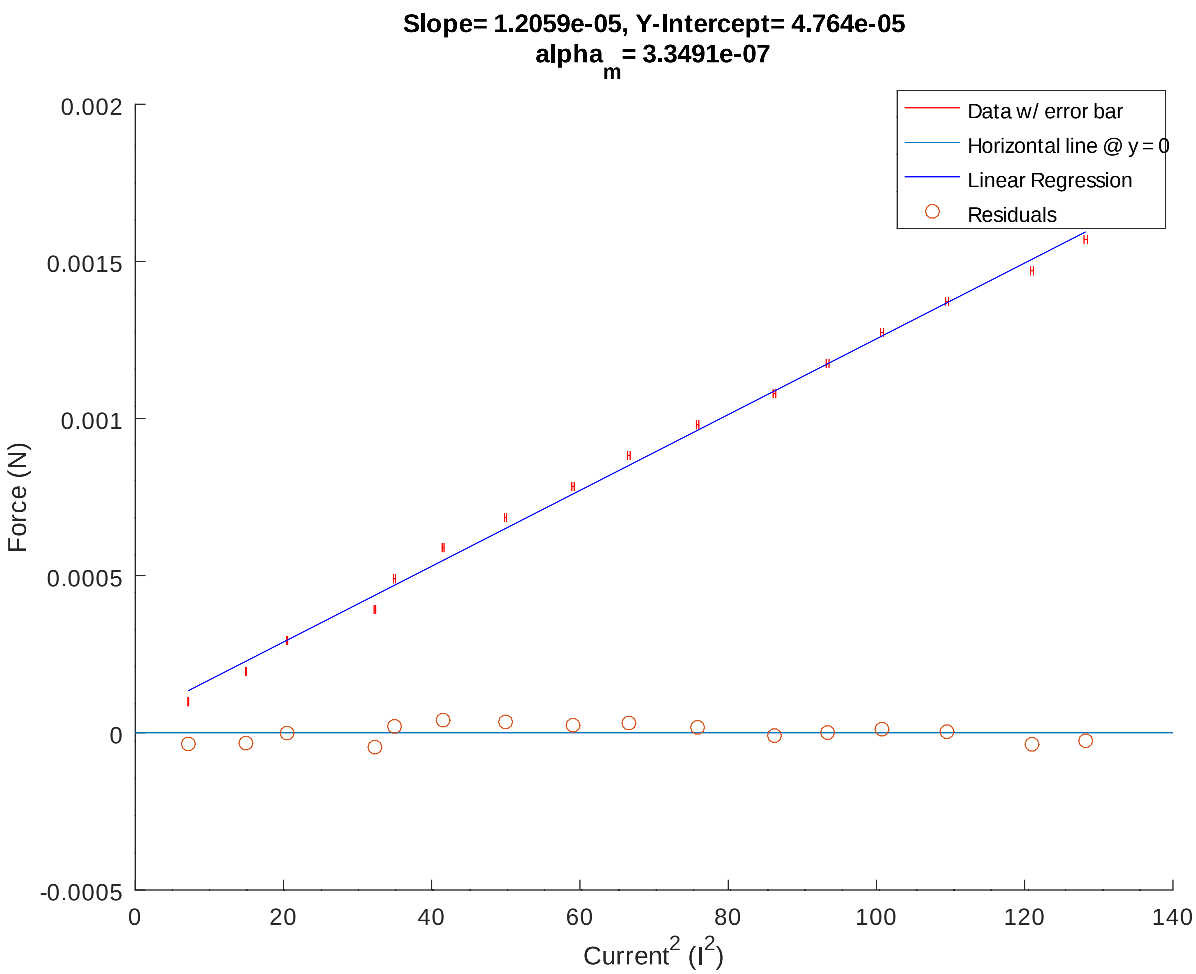

FIG. 4. This is a plot of the magnetic force (F) against the square of the current with appropriate stray-field corrections (I_+I_-). The residuals are also plotted near the horizontal axis.

Mass (mg)	Forward Current, I_+ (A)	Reverse Current, I_- (A)
10.1	3.37	2.68
19.9	4.42	3.74
30.0	5.58	4.79
40.0	6.76	5.77
50.0	7.67	6.46
60.1	8.42	7.31
69.9	8.87	7.82
80.0	9.53	8.45
90.0	9.90	8.85
100.0	10.66	9.47
110.1	11.08	10.11
119.9	11.72	10.56

TABLE III. The third data set of forward and reverse currents necessary to return the ruler reading to $0.0815m$ for varying masses.

By linear regression, the slope of the line of best-fit was calculated to be:

$$m = (9.4 \pm 0.2) \times 10^{-6} N/A^2.$$

Rearranging our expression for the slope, we obtain:

$$\begin{aligned}\frac{\mu_0}{2\pi} &= \frac{md}{L} \\ &= \frac{9.4 \times 10^{-6} N/A^2 \times 5.15 \times 10^{-3} m}{0.2760m} \\ &= (1.75 \pm 0.08) \times 10^{-7} N/A^2.\end{aligned}$$

Finally, comparing this to the actual value of $\mu_0/2\pi = 2 \times 10^{-7} N/A^2$, we obtain our percentage error.

$$\begin{aligned}\%_e &= \frac{X_{\text{exp}} - X_{\text{the}}}{X_{\text{the}}} \times 100\% \\ &= \frac{1.75 \times 10^{-7} N/A^2 - 2 \times 10^{-7} N/A^2}{2 \times 10^{-7} N/A^2} \times 100\% \\ &= 12.51\%.\end{aligned}$$

Again, we qualitatively see no patterns in the residuals, which means that the $\mu_0/2\pi$ estimates do not show a statistical change as we increase the masses being placed.

IV. CONCLUSION AND FUTURE WORK

In this experiment, we were able to measure the permeability of free space, μ_0, to a reasonably degree of accuracy: 10.85%, 8.06%, and 12.51% error for the three data sets. The results of $\mu_0/2\pi$ estimates were $1.78 \times 10^{-7} N/A^2$, $1.84 \times 10^{-7} N/A^2$, and $1.75 \times 10^{-7} N/A^2$ respectively. We derived the formula for the magnetic force between current-carrying wires using Ampere's Law, and applied the formula to devise a linear relationship between the magnetic force F and the squared currents I^2. To correct for the stray magnetic fields, we replaced I^2 with I_+I_-, where I_+ and I_- are currents measured when the circuit was ran in the opposite direction. As can be seen from the three data sets' plots of F against I_+I_-, our results are reasonably precise. For instance, the uncertainty in our estimate of $\mu_0/2\pi$ obtained from our first data set is only $0.07 \times 10^{-7} N/A^2$. In all three data sets, the true value of $2 \times 10^{-7} N/A^2$ is not included in our uncertainty range, which implies the presence of systematic errors. We shall carefully evaluate the experimental set-up and attempt to remove significant sources of systematic error in future iterations of this same experiment.

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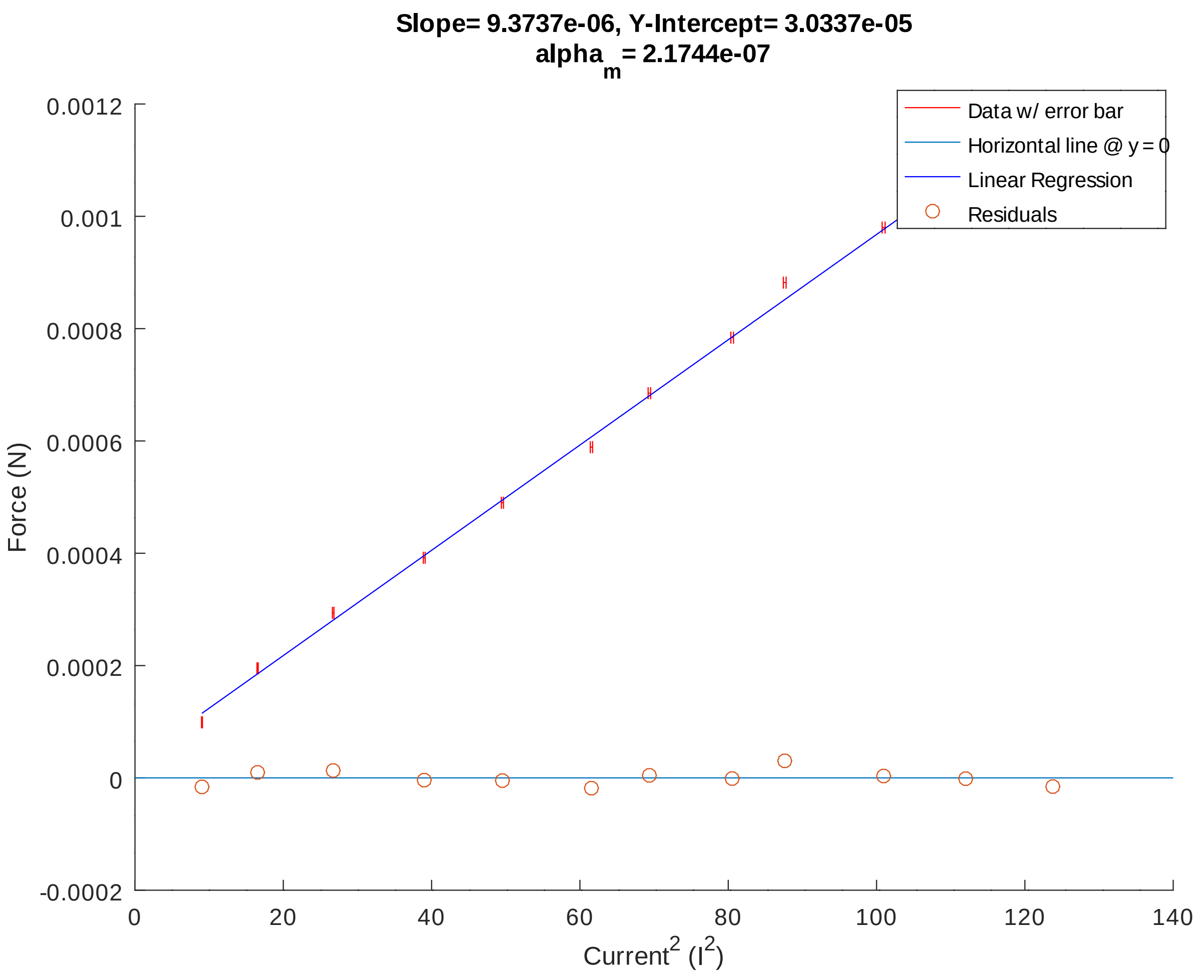

FIG. 5. This is a plot of the magnetic force (F) against the square of the current with appropriate stray-field corrections (I_+I_-). The residuals are also plotted near the horizontal axis.