Quantum Mechanics Research Fall 2018 Report

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1 Introduction

Ultimately, our objective is to come up with a more physically intuitive picture of quantum mechanics that does not rely on the mysterious complex function we call the wave-function. There are a couple alternative formalisms we can use to talk about quantum mechanics, and the Wigner-Weyl formalism that relies on Wigner functions (and thus bypassing the use of wave-functions) is one of them. Some immediate advantages of Wigner functions over wave-functions are that they are real-valued functions defined on phase space (functions of momentum and position), and that they are normalized, just like a probability distribution. Written mathematically,

$$f_W(p,x) \in \mathbb{R}, \text{ and}$$

$$\int f_W(p,x)\ dp\ dx = 1.$$

More properties of Wigner functions will be discussed later. If Wigner functions were probability distributions, or if we can somehow connect Wigner functions to probability distributions, that would be ideal since we understand probability distributions well, and that would provide a more physically intuitive picture of quantum mechanics. One complication is that Wigner functions can (and do) take on negative values, and we do not have a standard interpretation for negative probabilities if we were to interpret the Wigner functions directly as probability distributions. One idea that we, and other researchers, have been working on is to interpret the Wigner function as a current rather than a probability density. A good analogy is to think of river flow. Overall, the water will flow in one preferred direction, but of course it is entirely possible for some parts of the river to have back flows. In a similar manner, if we interpret the Wigner function as a current, we can think of the negative areas as back flows. Out of all the physical quantities available to us, the only one we currently do not consider directionality is time, and therefore, our current endeavor is in trying to come up with an interpretation of Wigner functions as currents in time, allowing quantum particles to travel backwards as well as forwards in time.

2 The Hamiltonian of a Free Particle Introducing a New Parameter τ

One of the many approaches we have attempted is to start with classical mechanics, and introduce a new parameter τ in order to allow back flows in time. The question is: can we make sense of simple systems we study in mechanics using this new mathematical set-up?

2.1 Basic Hamiltonian Definitions with the Parameter τ

Inspired by standard Hamiltonian mechanics, we will have the Hamiltonian function as $H(p, \mu, x, t; \tau)$ where p, μ, x, t all depend on the parameter τ. In this case, we can have similar mathematical relationships as in standard Hamiltonian mechanics:

$$\frac{dx}{d\tau} = \mathring{x} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{d\tau} = \mathring{p} = -\frac{\partial H}{\partial x}$$

$$\frac{dt}{d\tau} = \mathring{t} = \frac{\partial H}{\partial \mu}$$

$$\frac{d\mu}{d\tau} = \mathring{\mu} = -\frac{\partial H}{\partial t}$$

2.2 Mass as Time-momentum

In trying to study the free particle, let us initially assume a Hamiltonian of the form: $H = f(p^2 + \mu^2)$ that is independent of position (and thus any form of potential). Then, applying our definitions from above, we have:

$$\frac{dx}{d\tau} = \frac{\partial H}{\partial p} = f'(p^2 + \mu^2)2p$$

$$\frac{dt}{d\tau} = \frac{\partial H}{\partial \mu} = f'(p^2 + \mu^2)2\mu.$$

Using these results,

$$\frac{dx}{dt} = \frac{dx/d\tau}{dtd\tau} = \frac{f'2p}{f'2\mu} = \frac{p}{\mu}.$$

Now, comparing this with the known velocity of a free particle given by $v = p/m$, we can see that this agrees with standard classical mechanics if $\mu = m$. This implies that we might be interested in interpreting the mass of a particle as some sort of **time-momentum**.

3 The Hamiltonian of the Simple Harmonic Oscillator (SHO)

Another system for which we have attempted to apply this new mathematical set-up with parameter τ is the simple harmonic oscillator (SHO). Basically, we are interested in writing down a Hamiltonian function with the parameter τ that produces the familiar SHO behavior.

3.1 Experimenting with the Mathematics

One approach is to make up some Hamiltonian function and see if it works out. So here are a few note-worthy mathematical attempts in trying to produce SHO behavior.

3.1.1 $H = f(p^2 + m^2 + x^2)$

If we use a Hamiltonian that looks like $H = f(p^2 + m^2 + x^2)$, the partial derivatives don't work out.

$$\frac{dx}{d\tau} = \frac{\partial H}{\partial p} = f'2p$$
$$\frac{dt}{d\tau} = \frac{\partial H}{\partial m} = f'2m$$
$$\frac{dp}{d\tau} = -\frac{\partial H}{\partial x} = -f'2x$$
$$\frac{dm}{d\tau} = -\frac{\partial H}{\partial t} = 0$$

$$\implies \frac{dx}{dt} = \frac{p}{m}, \text{ and } \frac{dp}{dt} = -\frac{x}{m}.$$
$$\implies \ddot{x} = \frac{\dot{p}}{m} = -\frac{x}{m^2}.$$

This m-dependence is a problem, as it does not agree with the SHO behavior we study in standard classical mechanics.

3.1.2 $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$

If we use the standard Hamiltonian for a SHO $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, we get:

$$\mathring{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\mathring{t} = \frac{\partial H}{\partial m} = -\frac{p^2}{2m^2} + \frac{1}{2}\omega^2 x^2$$
$$\mathring{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

$$\mathring{m} = -\frac{\partial H}{\partial t} = 0.$$

So this at least gives us SHO behavior in τ, which might be worth exploring:

$$\mathring{\mathring{x}} = \frac{\mathring{p}}{m} = -\omega^2 x.$$

3.1.3 $H = f(g(p^2 + m^2) + h(x^2))$

If we use a Hamiltonian of form $H = f(g(p^2 + m^2) + h(x^2))$,

$$\frac{dx}{d\tau} = \frac{\partial H}{\partial p} = f'(g(p^2 + m^2) + h(x^2))g'(p^2 + m^2)2p$$

$$\frac{dt}{d\tau} = \frac{\partial H}{\partial m} = f'(g(p^2 + m^2) + h(x^2))g'(p^2 + m^2)2m$$

$$\frac{dp}{d\tau} = -\frac{\partial H}{\partial x} = -f'(g(p^2 + m^2) + h(x^2))h'(x^2)2x.$$

This gives us $\frac{dx}{dt} = \frac{p}{m}$ as desired, and:

$$\frac{dp}{dt} = -\frac{h'(x^2)}{g'(p^2 + m^2)}\frac{x}{m}.$$

So, this works if $\frac{h'(x^2)}{g'(p^2 + m^2)} = m$. So far, we have not been able to find a function that satisfies this relationship.

3.2 Using the Foundation of Lagrangian/Hamiltonian Formalism of Classical Mechanics

Another approach is to take a closer look at **why** the Lagrangian/Hamiltonian formalism of classical mechanics works in the first place. In other words, revisiting the whole action minimization business. Perhaps we might be able to form a mathematical argument that allows us to go through the action minimization derivation with our new parameter τ that will give us some insight into what the Hamiltonian function should look like. So far, I have read the derivations of the Lagrangian and Hamiltonian functions in several classical mechanics textbooks. I hope to insert my notes from those readings here soon.

4 Notes from Quantum Mechanics in Phase Space - Zachos, Fairlie, and Curtright

Now, another entirely different approach is to play around with Wigner functions instead of trying to make classical mechanics work. However, before we do that it is important that we familiarize ourselves with Wigner functions. Here are some of my most pertinent notes from Quantum Mechanics in Phase Space by Zachos, Fairlie, and Curtright.

- There are three logically autonomous formalisms of QM:
  1. Operators in Hilbert space; wave-functions; Heisenberg, Schrödinger, Dirac.
  2. Path integrals; Dirac, Feynman.
  3. Phase-space formulation; Wigner's quasi-distribution function, Weyl's correspondence between QM operators in Hilbert space and ordinary c-number functions in phase space, the $\star$-product: Groenewold and Moyal.
- Wigner functions are quasi-probability distribution functions in phase space.
$$f(x,p) = \frac{1}{2\pi}\int dy\ \Psi^*\left(x - \frac{\hbar}{2}y\right) e^{-iyp}\ \Psi\left(x + \frac{\hbar}{2}y\right)$$
- Wave-functions may be bypassed in principle.
- Negative probability back flows.
- Normalized: $\int\ dp\ dx\ f(x,p) = 1$.
- $f(x,p)$ is real.
- There is an x-p symmetry.
- In the classical limit $\hbar \to 0$, it reduces to a probability density.
- The Schwarz inequality: $-\frac{2}{h} \leq f(x,p) \leq \frac{2}{h}$.
- p- or x-projection leads to marginal probability densities:
  - space-like shadow: $\int dp\ f(x,p) = \rho(x)$.
  - momentum-space shadow: $\int dx\ f(x,p) = \sigma(p)$.
- $f(x,p)$ can be negative.
- $\langle \mathscr{G} \rangle = \int dx\ dp\ f(x,p)\ g(x,p)$ where kernel function $g(x,p)$ is often the unmodified classical observable.
- Moyal's equation (extension of Liouville's theorem of classical mechanics):
$$\frac{\partial f}{\partial t} = \frac{H \star f - f \star H}{i\hbar} \equiv \{\{H, f\}\},$$
where the $\star$-product is given by:
$$\star \equiv e^{\frac{i\hbar}{2}\left(\overleftarrow{\partial}_x \vec{\partial}_p - \overleftarrow{\partial}_p \vec{\partial}_x\right)}$$
In practice, we usually use:
$$f(x,p) \star g(x,p) = f\left(x + \frac{i\hbar}{2}\vec{\partial}_p, p - \frac{i\hbar}{2}\vec{\partial}_x\right) g(x,p).$$

- Static Wigner function obeys the $\star$-genvalue equations (akin to eigenvalue equations):
$$H(x,p) \star f(x,p) = f(x,p) \star H(x,p) = Ef(x,p),$$
where E is the energy eigenvalue of $H\Psi = E\Psi$.

5 Wigner Function for a Free Particle

Once again, we would like to study the simplest systems for which we know their Wigner function solutions. I undertook the task to play around with the Wigner function solution for a free particle, while Francesco Setti worked on the SHO solutions. The Wigner function solution for a free particle looks like:

$$f(x,p) = A\delta(p+\sqrt{2mE})+B\delta(p-\sqrt{2mE})+\delta(p)\left[Y\cos\left(\sqrt{\frac{8mE}{\hbar^2}}x\right)+Z\sin\left(\sqrt{\frac{8mE}{\hbar^2}}x\right)\right].$$

Read Equation of Motion of a Free particle for a derivation of this Wigner function. This essentially tells us that there are only three allowed values for the momentum: 0, and $\pm\sqrt{2mE}$. While the positive and negative momentum solutions essentially have just constant coefficients, the oscillatory behavior multiplying the zero momentum solution is very interesting as it inevitably carries parts of the Wigner function that would give us negative values.

Now, returning to the river flow analogy, water has a density, ρ, that is related to the currents, j_i's in the river by the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0.$$

This is just a mathematical expression of common sense. Changes in the density must be due to the inflow or outflow of the currents. So, our next objective is to construct a density function that satisfies the continuity equation when we take the Wigner function to be the time-component of the current, j_t.

5.1 Concentrating on the Zero-Momentum Solution

So the zero-momentum term of the Wigner function omitting the Dirac delta function looks like:
$$f(x) = Y\cos\left(\sqrt{\frac{8mE}{\hbar^2}}x\right)+Z\sin\left(\sqrt{\frac{8mE}{\hbar^2}}x\right).$$

We would first like to interpret this Wigner function as the time-component of the probability current density, j_t, and now we want to construct a probability density that is non-negative and satisfies the continuity equation:

$$\frac{\partial \mathbb{P}}{\partial \tau} + \frac{\partial}{\partial t}j_t + \frac{\partial}{\partial x}j_x = 0.$$

So far, here is our best attempt. To begin, we may picture the zero-momentum solution as circles or loops in the x-t plane since our zero-momentum solution is sinusoidal. Now, if we set $x = A\cos(ct) + B\sin(ct)$, that is consistent with our picture that the zero-momentum solutions are indeed circles in the x-t plane. In the figure below, the horizontal sinusoids are functions of form $t = C\cos(dx) + D\sin(dx)$, the vertical sinusoids are functions of form $x = A\cos(ct) + B\sin(ct)$, and we notice that those components of currents are consistent with our circular loop picture, shown in red. Furthermore, it also satisfies the continuity equation, since $\frac{\partial}{\partial t} j_t = 0$ and $\frac{\partial}{\partial x} j_x = 0$, as long as $\frac{\partial \mathbb{P}}{\partial \tau} = 0$ which means that $\mathbb{P}$ is independent of τ.