Measures and Hausdorff Dimension

Tatsuya Akiba

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1 Introduction

One of the major topics covered in a standard, introductory-level calculus sequence is the Riemann integral. Specifically, we learn that a definite integral of a function gives us the area under the curve. As such, the Riemann integral is geometrically very appealing and finds many applications in fields such as physics, probability, and economics. Furthermore, the Riemann integral works brilliantly for continuous, bounded functions and suffices in most situations (at least as an approximation) for these applied fields of study. Nonetheless, mathematically, the Riemann integral is limited. From the geometric interpretation of the integral as the area under the curve, one can easily see the challenge it faces when given an unbounded function. Additionally, the Riemann integral fails to deal with general bounded functions as we introduce an infinitude of discontinuities. The canonical example of such a general bounded function is the Dirichlet function.

Example 1.1 (Dirichlet Function)**.** *The Dirichlet function, $f : \mathbb{R} \to \mathbb{R}$, is defined as follows:*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Claim 1.2. *The Dirichlet function is not Riemann integrable.*

Proof. Let $a, b \in \mathbb{R}$ such that $b > a$, and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$.

Note that for every interval $[x_{i-1}, x_i]$ in the partition P, there exists $p_i \in \mathbb{Q}$ such that $x_{i-1} < p_i < x_i$ and there also exists $q_i \in \mathbb{R} \setminus \mathbb{Q}$ such that $x_{i-1} < q_i < x_i$ since $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense.

Thus,

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} = 1$$

and

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0$$

for every interval in the partition.

Now, the upper Riemann sum of f with respect to P is:

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (x_i - x_{i-1}) = b - a, \end{aligned}$$

and the lower Riemann sum of f with respect to P is:

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 0.$$

Since this is true of all partitions, the upper Riemann integral of f is:

$$U(f) = \inf\{U(f,P) : \text{P is a partition of } [a,b]\} = b - a$$

and the lower Riemann integral of f is:

$$L(f) = \sup\{L(f,P) : \text{P is a partition of } [a,b]\} = 0.$$

Therefore, since $U(f) \neq L(f)$, the Dirichlet function is not Riemann integrable. ■

Although the range of the Dirichlet function is extremely simple, only taking on the values 0 and 1, the Riemann integral fails to deal with the complexity of the sets used to define the function, namely $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$, and the discontinuities they cause. If we instead consider a similar function defined in terms of intervals, we notice that it is quite easily Riemann integrable. For instance, let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition on some interval $[a,b]$ and let the function g be defined by:

$$g(x) = \begin{cases} 1, & x \in [x_{i-1}, x_i) \text{ where i is even} \\ 0, & x \in [x_{i-1}, x_i) \text{ where i is odd.} \end{cases}$$

For g, we can easily show that the integral is given by:

$$\int_a^b g(x)dx = \sum_{\text{i is even}}^{n} (x_i - x_{i-1}).$$

What, then, prevents us from integrating the Dirichlet function? By comparing the way the Dirichlet function is defined against similar, but integrable functions such as g defined above, one might suspect that the reason we fail to integrate the Dirichlet function is because we have not yet introduced a way to measure sets like $\mathbb{Q}$ or $\mathbb{R} \setminus \mathbb{Q}$ in the same way we measure intervals using lengths.

It turns out that this is a good guess. By generalizing the notion of length for intervals to more complicated sets, we can successfully formulate an integral that remarkably improves the variety of functions we are able to integrate. In this paper, we will not be discussing integration any further, although there is plenty to be discussed about there. Rather, we shall focus on the notion of measure, the generalization of length, as it is mathematically intriguing in its own right, and present further interesting mathematics derived from measure theory.

2 General Measure Theory

We begin with the general definition of outer measure which gives us some of the basic structure of a length-like function extended to all subsets of a set X. While we will later look at a very specific outer measure defined on $\mathbb{R}$ in Section 3.1, we give a definition that applies to any set X here in this section.

Definition 2.1. Let X be a nonempty set. An **outer measure** μ on a set X is a function defined on all subsets of X taking values in $[0, \infty]$ such that

1. $\mu(\emptyset) = 0$,
2. $\mu(A) \leq \mu(B)$ if $A \subseteq B$ (monotonic), and
3. $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$ for any collection of subsets $\{A_n\}_{n=1}^{\infty}$ of X (countably subadditive).

Although this is a good start, we desire a little bit more structure, at least in the context of extending the notion of length in $\mathbb{R}$. Specifically, with this definition of an outer measure, we do not have the convenient property of countable additivity. In other words, it is not necessarily the case that $\mu(A \cup B) = \mu(A) + \mu(B)$ even when A and B are disjoint sets. In order to define a measure for which we have this property, we first consider the collections of sets on which a measure can be defined.

Definition 2.2. Let X be a nonempty set. A collection $\mathscr{M}$ of subsets of X is called a **σ-algebra** of sets in X if

1. $X \in \mathscr{M}$;
2. $A \setminus B \in \mathscr{M}$ whenever $A, B \in \mathscr{M}$;
3. $\bigcup_{n=1}^{\infty} A_n = \mathscr{M}$ whenever $\{A_n\}_{n=1}^{\infty}$ is in $\mathscr{M}$.

For concreteness, we include an example.

Example 2.3. *Let X be any set. Then, the power set of X, $\mathscr{P}(X)$, is a σ-algebra.*

It is trivial to check that $\mathscr{P}(X)$ satisfies the three conditions in Definition 2.2. Now, we define the notion of measure for a general σ-algebra.

Definition 2.4. Let $\mathscr{M}$ be a σ-algebra in X. A **positive measure** on $\mathscr{M}$ is a function $\mu : \mathscr{M} \to [0, \infty]$ which satisfies

1. $\mu(\emptyset) = 0$;

2. μ is countably additive: If $\{A_n\}_{n=1}^{\infty}$ in $\mathscr{M}$ is a pairwise disjoint collection of sets, then
$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The triple $(X, \mathscr{M}, \mu)$ is called a **measure space** on X.

Although we have introduced and often think about measure as the extension of length, we are not ready to develop such a measure at this time. For now, we give a brief example of a measure which, instead of length, gives us the size (the number of elements) of a set.

Example 2.5 (Counting Measure)**.** *Let $X = \mathbb{N}$ and set $\mathscr{M} = \mathscr{P}(\mathbb{N})$. Define $\mu : \mathscr{P}(\mathbb{N}) \to [0, \infty]$ by*
$$\mu(A) = \begin{cases} n(A), & A \text{ is finite;} \\ \infty, & A \text{ is infinite} \end{cases}$$
where $n(A)$ denotes the number of elements in A.

It is relatively straightforward to prove that this definition indeed gives a measure on $\mathscr{P}(\mathbb{N})$. We immediately see that $\mu(\emptyset) = n(\emptyset) = 0$, satisfying the first property of 2.4, and for any disjoint sets A and B, $\mu(A \cup B) = n(A \cup B) = n(A) + n(B) = \mu(A) + \mu(B)$, which generalizes to satisfy property 2.

We now list some useful properties of measures, some of which we have encountered before when we defined a general outer measure in Definition 2.1.

Theorem 2.6 (Properties of Measures)**.** *Let $\mathscr{M}$ be a σ-algebra on a set X and let μ be a positive measure on $\mathscr{M}$.*

1. *Monotonicity - If $A, B \in \mathscr{M}$ with $A \subseteq B$, then $\mu(A) \leq \mu(B)$.*

2. *Continuity for increasing unions - Suppose $\{A_n\}_{n=1}^{\infty}$ is in $\mathscr{M}$ with $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and $A = \bigcup_{n=1}^{\infty} A_n$. Then*
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

3. *Continuity for decreasing intersections - Suppose $\{A_n\}_{n=1}^{\infty}$ is in $\mathscr{M}$ with $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and $A = \bigcap_{n=1}^{\infty} A_n$. If $\mu(A_1) < \infty$, then*
$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

4. *Countably subadditive - If $\{A_n\}_{n=1}$ is in $\mathscr{M}$, then $\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.*

Proof.

1. Let $A, B \in \mathscr{M}$ and $A \subseteq B$. Note that $B = A \cup (B \setminus A)$, which is a disjoint union of sets in $\mathscr{M}$. By the additivity of μ, we have:

$$\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A) \geq \mu(A).$$

2. Let $\{A_n\}_{n=1}^{\infty}$ be a collection of sets in $\mathscr{M}$ such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and let $A = \bigcup_{n=1}^{\infty} A_n$. Now, define a collection of sets $\{B_i\}_{i=1}^{\infty}$ by $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$ for $i = 2, 3, 4, \ldots$ such that $\{B_i\}$ is a pairwise disjoint collection of sets in $\mathscr{M}$. Note that $A_n = \bigcup_{i=1}^{n} B_i$ and thus $A = \bigcup_{n=1}^{\infty} B_i$. Then, by the additivity of μ, we have:

$$\mu(A) = \sum_{i=1}^{\infty} \mu(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \mu(B_i) = \lim_{n\to\infty} \mu(A_n).$$

3. Let $\{A_n\}_{n=1}^{\infty}$ be a collection of sets in $\mathscr{M}$ such that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and let $A = \bigcap_{n=1}^{\infty} A_n$. Assume $\mu(A_1) < \infty$. Similar to (2), define a collection of sets $\{C_i\}_{i=1}$ by $C_i = A_1 \setminus A_i$. First, note that $C_1 \subseteq C_2 \subseteq C_3 \subseteq \ldots$, so we can use our result from (2). Also observe that, since $A_1 = A_n \cup (A_1 \setminus A_n)$, we can write $\mu(A_1) = \mu(A_n) + \mu(A_1 \setminus A_n)$ by countable additivity. Thus, $\mu(C_n) = \mu(A_1 \setminus A_n) = \mu(A_1) - \mu(A_n)$. Finally we note that:

$$\begin{aligned} A_1 \setminus A = A_1 \setminus \bigcap_{n=1}^{\infty} A_n &= A_1 \cap \overline{\bigcap_{n=1}^{\infty} A_n} = A_1 \cap \bigcup_{n=1}^{\infty} \overline{A_n} \\ &= \bigcup_{n=1}^{\infty} A_1 \cap \overline{A_n} = \bigcup_{n=1}^{\infty} A_1 \setminus A_n = \bigcup_{n=1}^{\infty} C_n. \end{aligned}$$

Then,

$$\begin{aligned} \mu(A_1 \setminus A) &= \mu\left(\bigcup_{n=1}^{\infty} C_n\right) \\ &= \lim_{n\to\infty} \mu(C_n), \text{ by our result from (2)} \\ &= \lim_{n\to\infty} (\mu(A_1) - \mu(A_n)) \\ &= \mu(A_1) - \lim_{n\to\infty} \mu(A_n). \end{aligned}$$

Since we can also write $\mu(A_1 \setminus A) = \mu(A_1) - \mu(A)$ and $\mu(A_1)$ is finite by assumption, we immediately get the desired result:

$$\mu(A) = \lim_{n\to\infty} \mu(A_n).$$

4. We will prove this by induction. Let $\{A_n\}_{n=1}^{\infty}$ be a collection of sets in $\mathscr{M}$. For the base case, we note that $\mu\left(\bigcup_{n=1}^{2} A_n\right) = \mu(A_1) = \sum_{n=1}^{1} \mu(A_n)$, satisfying

the inequality. Now, assume that the result holds for some integer $k \geq 1$, so $\mu\left(\bigcup_{n=1}^{k} A_n\right) \leq \sum_{n=1}^{k} \mu(A_n)$. Then, for $k+1$,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{k+1} A_n\right) = \mu\left(\bigcup_{n=1}^{k} A_n \cup A_{k+1}\right) &= \mu\left(\bigcup_{n=1}^{k} A_n \cup \left(A_{k+1} \setminus \bigcup_{n=1}^{k} A_n\right)\right) \\
&= \mu\left(\bigcup_{n=1}^{k} A_n\right) + \mu\left(A_{k+1} \setminus \bigcup_{n=1}^{k} A_n\right) \\
&\leq \sum_{n=1}^{k} \mu(A_n) + \mu(A_{k+1}) = \sum_{n=1}^{k+1} \mu(A_n).
\end{aligned}$$

By the principle of mathematical induction, the result holds for any positive integer. Finally, we use our result from (2) to state that:

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} A_n\right) \\
&\leq \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n) = \sum_{n=1}^{\infty} \mu(A_n). \quad \blacksquare
\end{aligned}$$

Now, while an outer measure can be defined on all subsets of X, a measure can only be defined on a σ-algebra $\mathscr{M}$ of subsets of X. Although we will follow a different construction of a measure in $\mathbb{R}$ in Section 3, we outline a simple criterion used to restrict the subsets of X for which an outer measure recovers the property of countable additivity.

Definition 2.7 (Carathéodory's Criterion)**.** Let X be a nonempty set and μ be an outer measure on X. A subset A of X is called **μ-measurable** if it decomposes every subset of X additively, that is, if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A)$$

for all $B \subseteq X$.

This criterion will become important in discussing Hausdorff measure in Section 4.

It is important to note that the results discussed in this section are applicable to any general measure space $(X, \mathscr{M}, \mu)$. However, for our purposes, we will focus on Euclidean space, $\mathbb{R}^n$, and devote most of our attention to the simplest $\mathbb{R}^1$.

3 Constructing a Measure in $\mathbb{R}$

Now, we are ready to begin the construction of a measure that is the extension of length in $\mathbb{R}^1$. First, we will attempt to construct a measure on a simpler collection of sets and work our way up to more complicated subsets of $\mathbb{R}$.

Definition 3.1. Define the **elementary subsets** of $\mathbb{R}$ to be the subsets of $\mathbb{R}$ which are finite unions of disjoint, finite intervals. Denote the collection of all elementary subsets of $\mathbb{R}$ by $\mathscr{E}$.

Unfortunately, this is not a σ-algebra since taking the union of countably many elementary subsets may give rise to a set of infinite length. Such a set cannot be written as a finite union of finite intervals and is therefore not in $\mathscr{E}$. Nonetheless, for future purposes we will define a measure-like function, m, on the collection $\mathscr{E}$.

Definition 3.2. Define the function $m : \mathscr{E} \to [0, \infty)$ by

1. $m(\emptyset) = 0$;
2. if $A = \bigcup_{n=1}^{N} I_n \in \mathscr{E}$, a disjoint union of intervals, we set

$$m\left(\bigcup_{n=1}^{N} I_n\right) = \sum_{n=1}^{N} \ell(I_n),$$

   where $\ell(I_n)$ represents the length of I_n.

By definition, for any interval I, $m(I)$ gives us just the length of the interval. This is a good check, since in our construction we must make sure that our measure will be a genuine generalization of length. In other words, we require that our measure agrees with length on any interval.

For concreteness, we take a simple example.

Example 3.3. *Let $A = \bigcup_{n=1}^{N} [n, n+1)$.*
Clearly, $A \in \mathscr{E}$, and

$$\begin{aligned} m(A) &= m\left(\bigcup_{n=1}^{N} [n, n+1)\right) \\ &= \sum_{n=1}^{N} \ell([n, n+1)) = N. \end{aligned}$$

3.1 Outer Measure in $\mathbb{R}$

Our ultimate goal is to extend the measure-like function, m, from Definition 3.2 to more complex subsets of $\mathbb{R}$. Ideally, we would like to develop a generalization of length which preserves the length of an interval, the translation invariance of length, and the additive feature of length, but it turns out that we must sacrifice additivity in order to measure all subsets of $\mathbb{R}$. Accordingly, we introduce an outer measure in $\mathbb{R}$, which for our purposes will simply be an intermediary concept that will help us define a full-fledged measure in $\mathbb{R}$.

Definition 3.4. Let $A \subseteq \mathbb{R}$ and define an **outer measure** of A by

$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : I_n \text{ is an open interval and } A \subseteq \bigcup_{n=1}^{\infty} I_n\right\}.$$

Loosely speaking, we are covering the set A with open intervals in the most efficient manner possible, and the outer measure can be given as the sum of the lengths of such open intervals. To gain a more intuitive understanding of this definition, we give two examples.

Example 3.5. *Let $(a, b) \in \mathbb{R}$. Then, $m^*((a, b)) = b - a$.*

This example is fairly intuitive as you can cover (a, b) by taking one open interval: itself. An alternative way to produce the same result is by letting $\epsilon > 0$ and considering the interval $(a-\epsilon, b+\epsilon)$. For any ϵ, $(a, b) \subseteq (a-\epsilon, b+\epsilon)$ and the length of this interval is $(b-a)+2\epsilon$. Now, $\inf\{(b-a)+2\epsilon : \epsilon > 0\} = b-a$. However, this only shows that $m^*((a, b)) \leq b - a$. It turns out that $m^*((a, b)) = b - a$ as our intuition would suggest, but it takes a little more effort to prove this result rigorously. A more complete treatment of this example is left as an appendix (Appendix A).

Example 3.6. $m^*(\mathbb{Q}) = 0$.

In this example, we see a stark difference between length and outer measure. While it makes no sense to talk about the length of $\mathbb{Q}$, we can readily compute its outer measure by using our definition. Since $\mathbb{Q}$ is countable, let $\{q_1, q_2, q_3, \dots\}$ be an enumeration of $\mathbb{Q}$, and let $\epsilon > 0$. Then, define open intervals I_n by $I_n = (q_n - \frac{\epsilon}{2^{n+1}}, q_n + \frac{\epsilon}{2^{n+1}})$ so that $q_n \in I_n$ for all n. Since $\ell(I_n) = \frac{\epsilon}{2^n}$, we see that $\sum_{n=1}^{\infty} \ell(I_n) = \epsilon$. Therefore, $m^*(\mathbb{Q}) = \inf\{\epsilon : \epsilon > 0\} = 0$.

Now, we summarize some useful properties of outer measure in the following theorem.

Theorem 3.7. *[Properties of Outer Measure on $\mathbb{R}$] Let A and B be subsets of $\mathbb{R}$.*

1. *$m^*(A)$ is defined, though the value may be infinite.*
2. *$m^*(\emptyset) = 0$ and $m^*(\{a\}) = 0$.*
3. *m^* preserves m - If $A \in \mathscr{E}$, then $m(A) = m^*(A)$, where m is our measure-like function from Definition 3.2.*
4. *m^* is monotonic - If $A \subseteq B$, then $m^*(A) \leq m^*(B)$.*
5. *m^* is countably subadditive - If $\{A_n\}_{n=1}$ is in $\mathscr{P}(\mathbb{R})$, then*

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n).$$

Proof. **refer to Pons** ■

Although the outer measure can be calculated for any subset of $\mathbb{R}$, we have lost the convenient property of additivity when constructing this generalization. In other words, we generalized too far. Our next goal is to define a collection $\mathscr{L}$ of subsets of $\mathbb{R}$ on which the additivity property is recovered, but before we define $\mathscr{L}$, we must take a quick detour.

Definition 3.8. Let $X, Y \subseteq \mathbb{R}$. Define a semi-metric $d : \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R}) \to [0, \infty]$ by

$$d(X, Y) = m^*(X \Delta Y),$$

where Δ denotes the symmetric difference of two sets, so $X \Delta Y = (X - Y) \cup (Y - X)$.

A semi-metric is a function $f : \mathscr{P}(\mathbb{R}) \times \mathscr{P}(\mathbb{R}) \to [0, \infty]$ that satisfies:

1. $f(X, X) = 0$,
2. $f(X, Y) = f(Y, X)$, and
3. $f(X, Z) \leq f(X, Y) + f(Y, Z)$

for any $X, Y, Z \subseteq \mathbb{R}$. A metric additionally satisfies the property that $f(X, Y) = 0 \iff X = Y$. Although the semi-metric d we defined above is not central to our discussion of measures, a clear understanding of its operation will become important as we move toward defining the Lebesgue-measurable sets. To get a better handle on this semi-metric, we provide a simple example.

Example 3.9. *Let $A = [0, 2]$, $B = [1, 3]$, and $C = (0, 2)$. Computing the symmetric differences,*

$$A \Delta B = [0, 1) \cup (2, 3], \textit{ and}$$
$$A \Delta C = \{0\} \cup \{2\}.$$

Thus,

$$d(A, B) = m^*(A \Delta B) \quad = m^*([0, 1) \cup (2, 3]) = 2, \textit{ and}$$
$$d(A, C) = m^*(A \Delta C) \quad = m^*(\{0\} \cup \{2\}) = 0.$$

In this example, take a moment to notice that although $A \neq C$, $d(A, C) = 0$. This explicitly shows that for d, it is not the case that $d(X, Y) = 0 \iff X = Y$ which is the reason d is a semi-metric and not a metric.

In this next proposition, we list some of the properties of this semi-metric d.

Proposition 3.10. *Let $A, B, C, D \subseteq \mathbb{R}$.*

1. $d(A, \emptyset) = m^*(A)$
2. $d(A, C) \leq d(A, B) + d(B, C)$ *(triangle inequality)*
3. $d(A \cup B, C \cup D) \leq d(A, C) + d(B, D)$
4. $d(A \setminus B, C \setminus D) \leq d(A, C) + d(B, D)$
5. *If $m^*(B) \leq \infty$, then $|m^*(A) - m^*(B)| \leq d(A, B)$.*

Proof. **refer to Pons** ∎

We are now ready to extend our collection of elementary subsets, $\mathscr{E}$, using our semi-metric d defined in Definition 3.8.

Definition 3.11. Let $\overline{\mathscr{E}}$ denote the collection of subsets $A \subseteq \mathbb{R}$ for which there exists a sequence (A_n) in $\mathscr{E}$ with $d(A_n, A) \to 0$ as $n \to \infty$.

Another way of looking at this definition is that we are now including all the limit points of $\mathscr{E}$ with respect to the semi-metric d. In this way, the "overline" notation in $\overline{\mathscr{E}}$ is appropriate as it is the closure of $\mathscr{E}$ with respect to d.

Although we have not yet defined a true measure, the next lemma will assert that additivity has been recovered for the outer measure in the collection $\overline{\mathscr{E}}$.

Lemma 3.12. *If $\{A_n\}_{n=1}^{\infty}$ in $\overline{\mathscr{E}}$ is a pairwise disjoint collection of sets and $A = \bigcup_{n=1}^{\infty} A_n$, then*

$$m^*(A) = \sum_{n=1}^{\infty} m^*(A_n).$$

Proof. We already have countable subadditivity for an outer measure by Property 5 of Theorem 3.7, so:

$$m^*(A) = m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^*(A_n).$$

To show the reverse inequality, we note that $\bigcup_{n=1}^{N} A_n \subseteq A$ for each $N \in \mathbb{N}$. Thus, by monotonicity (Property 4 of Theorem 3.7), we have:

$$\sum_{n=1}^{N} m^*(A_n) = m^*\left(\bigcup_{n=1}^{N} A_n\right) \le m^*(A) \text{ for each } N \in \mathbb{N}.$$

Thus,

$$\sum_{n=1}^{\infty} m^*(A_n) \le m^*(A).$$

Now we have shown the inequality in both directions. Therefore,

$$\sum_{n=1}^{\infty} m^*(A_n) = m^*(A).$$ ■

We first give a quick example of a set in $\overline{\mathscr{E}}$.

Example 3.13. *Let $A_n = \left(\frac{1}{2n}, \frac{1}{2n-1}\right)$, and define $\mathcal{A} = \bigcup_{n=1}^{\infty} A_n$.*

Looking at unions of some of the first sets in the sequence (A_n):

$$A_1 = \left(\frac{1}{2}, 1\right),$$

$$A_1 \cup A_2 = \left(\frac{1}{4}, \frac{1}{3}\right) \cup \left(\frac{1}{2}, 1\right), \text{ and}$$

$$A_1 \cup A_2 \cup A_3 = \left(\frac{1}{6}, \frac{1}{5}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \left(\frac{1}{2}, 1\right).$$

Clearly, each of these is a disjoint union of finite intervals, so $\bigcup_{n=1}^{N} A_n \in \mathscr{E}$ for any $N \in \mathbb{Z}^+$. Therefore, $\mathcal{A} \in \overline{\mathscr{E}}$ by definition. We have yet to prove that the outer measure in $\mathbb{R}$ recovers the property of additivity in this collection $\overline{\mathscr{E}}$, but let us go ahead and compute the outer measure of $\mathcal{A}$. By Theorem 3.7 Property 3, we know that for each $\bigcup_{n=1}^{N} A_n$,

$$m^*\left(\bigcup_{n=1}^{N} A_n\right) = m\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \ell\left(\left(\frac{1}{2n}, \frac{1}{2n-1}\right)\right) = \sum_{n=1}^{N}\left(\frac{1}{2n-1} - \frac{1}{2n}\right).$$

Therefore,

$$\begin{aligned} m^*(\mathcal{A}) &= \sum_{n=1}^{\infty}\left(\frac{1}{2n-1} - \frac{1}{2n}\right) \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \\ &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} \\ &= \ln(2), \end{aligned}$$

where for our final equality we have used the well-known Maclaurin series

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

for $x = 1$.

Next, we introduce an important set, central to our discussion from hereon, which also happens to be in $\overline{\mathscr{E}}$.

Example 3.14 (Cantor Ternary Set). *Let $C_0 = [0,1]$, and define a sequence in $\mathscr{E}$ by*

$$C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right) \text{ for } n \geq 1.$$

Then, the Cantor ternary set, $\mathcal{C}$, can be defined by $\mathcal{C} = \bigcap_{n=1}^{\infty} C_n$.

As illustrated in Figure 1, we can gain an intuitive understanding of the Cantor ternary set as follows. We begin with the interval $C_0 = [0,1]$, and remove the middle third (keeping the endpoints), so that the remaining set is

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Now, we remove the middle third from each of the intervals in C_1 to get

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

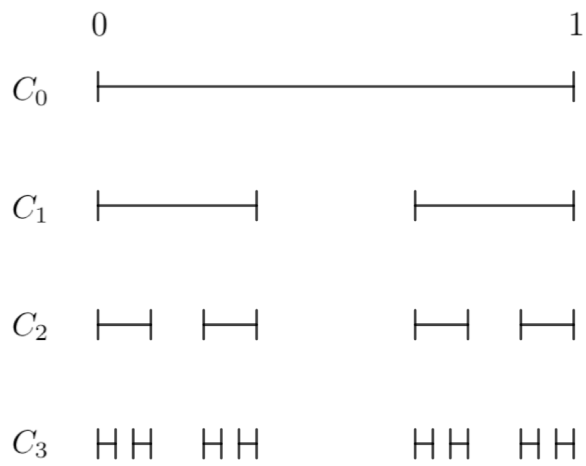

Figure 1: The First Four Sets in the Sequence C_n.

The remaining set after repeating this process infinitely many times is the Cantor ternary set. Once again, let us calculate the outer measure of this set. Note that each $C_n \in \mathscr{E}$. Also notice that since we are removing a third of each interval every time, the outer measure of C_n for each n is given by

$$m^*(C_n) = \left(\frac{2}{3}\right)^n.$$

Then, since $\mathcal{C} \subseteq C_n$ for all n,

$$m^*(\mathcal{C}) \leq m^*(C_n) = \left(\frac{2}{3}\right)^n$$

for all n by monotonicity. Therefore, we see that $\mathcal{C} \in \overline{\mathscr{E}}$, and

$$m^*(\mathcal{C}) = 0.$$

We will return to this example in Section 4 once we have developed further mathematical apparatus to analyze the true complexity of the Cantor ternary set, but for now, we shall complete our construction of a measure in $\mathbb{R}$.

3.2 Lebesgue Measure

We are finally ready to define the collection $\mathscr{L}$ for which the outer measure is a legitimate measure which satisfies all the properties of Theorem 2.6.

Definition 3.15. Let $\mathscr{L}$ denote the collection of subsets of $\mathbb{R}$ which can be written as a countable union of sets in $\overline{\mathscr{E}}$.

First, let us investigate what new sets have been added to this collection as compared to $\overline{\mathscr{E}}$. Recall that since $\mathscr{E}$ only included finite unions of finite intervals, $\overline{\mathscr{E}}$ was still limited to subsets of $\mathbb{R}$ with finite outer measure. With our new definition of $\mathscr{L}$, we have added the subsets of $\mathbb{R}$ with infinite outer measure. Since $\overline{\mathscr{E}}$ already included all the limit points of $\mathscr{E}$ with respect to the semi-metric d, we might guess that those subsets with infinite outer measure are the only ones added by this extension to $\mathscr{L}$. This guess turns out to be correct.

Proposition 3.16. *Let $A \in \mathscr{L}$. Then $A \in \overline{\mathscr{E}}$ if and only if $m^*(A) < \infty$.*

Proof. First, to prove the forward direction, suppose $A \in \overline{\mathscr{E}}$. By definition, there exists a sequence (A_n) in $\mathscr{E}$ with $d(A, A_n) \to 0$ as $n \to \infty$. For such a sequence, we can choose an $N \in \mathbb{N}$ with $d(A, A_N) < 1$, and recall that $m^*(A_N) = m(A_N) < \infty$. Now, by the triangle inequality (Proposition 3.10 (2)),

$$m^*(A) = m^*(A\Delta\emptyset) = d(A, \emptyset) \leq d(A, A_N) + d(A_N, \emptyset) < 1 + m^*(A_N) < \infty.$$

For the reverse direction, suppose $A \in \mathscr{L}$ with $m^*(A) < \infty$. By definition, there exists a collection of sets $\{A_n\}_{n=1}^{\infty}$ in $\overline{\mathscr{E}}$ with $A = \bigcup_{n=1}^{\infty} A_n$. Given such a collection $\{A_n\}$, define a pairwise disjoint collection of sets $\{B_n\}$ by:

$$B_1 = A_1, B_2 = A_2 \setminus A_1, B_3 = A_3 \setminus (A_1 \cup A_2),$$

so in general,

$$B_n = A_n \setminus (A_1 \cup A_2 \cup \cdots \cup A_{n-1}).$$

Since each of these sets is in $\overline{\mathscr{E}}$ and the B_n's are pairwise disjoint with $A = \bigcup_{n=1}^{\infty} B_n$, $\sum_{n=1}^{\infty} m^*(B_n) = m^*(A) < \infty$ by Lemma 3.12. Now, define another sequence by $C_n = B_1 \cup B_2 \cup \cdots \cup B_n \in \overline{\mathscr{E}}$. For each $m \in \mathbb{N}$, choose a set $D_m \in \mathscr{E}$ so that $d(C_m, D_m) < 1/m$ which is possible since each $C_m \in \overline{\mathscr{E}}$. Now, let $\varepsilon > 0$. Then, choose $M_1 > 0$ so that $1/M_1 < \varepsilon/2$ and choose $M_2 > 0$ such that

$$\sum_{n=M_2}^{\infty} m^*(B_n) < \varepsilon/2,$$

and set $m > \max\{M_1, M_2\}$. Then,

$$\begin{aligned} d(A, D_m) &\leq d(A, C_m) + d(C_m, D_m) = d\left(A, \bigcup_{n=1}^{m} B_n\right) + d(C_m, D_m) \\ &= d\left(\bigcup_{n=1}^{\infty} B_n, \bigcup_{n=1}^{m} B_n\right) + d(C_m, D_m) = m^*\left(\bigcup_{n=m+1}^{\infty} B_n\right) + d(C_m, D_m) \\ &= \left(\sum_{n=m+1}^{\infty} m^*(B_n)\right) + d(C_m, D_m) \leq \left(\sum_{n=M_2}^{\infty} m^*(B_n)\right) + 1/m \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, $d(A, D_m) \to 0$ as $m \to \infty$, so $A \in \overline{\mathscr{E}}$. ∎

By our definition, we must have a σ-algebra before we can define a measure. Our next theorem asserts that $(\mathbb{R}, \mathscr{L}, m^*)$ forms a measure space.

Theorem 3.17. *The collection $\mathscr{L}$ is a σ-algebra of sets in $\mathbb{R}$ and m^* is a measure on $\mathscr{L}$.*

Proof. **to be done later** ■

We have now successfully generalized length in $\mathbb{R}$ by discovering a collection of subsets of $\mathbb{R}$ which forms a σ-algebra together with a measure on the collection with all the convenient properties of Theorem 2.6.

Definition 3.18. We call the collection $\mathscr{L}$ the **Lebesgue measurable subsets** of $\mathbb{R}$ and the measure defined by $m(A) = m^*(A)$ for $A \in \mathscr{L}$ the **Lebesgue measure** on $\mathbb{R}$.

include Caratheodory's criterion proof or discussion from paper

Finally, we give a simple yet revealing example which illustrates the significant improvement we've made by extending the notion of length and recovering the property of additivity.

Example 3.19. *Let $A = (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1]$.*

In order to compute $m(A)$, first recall that $m(\mathbb{Q}) = 0$ from Example 3.6. Now, let $B = \mathbb{R} \cap [0, 1] = [0, 1]$ and $C = \mathbb{Q} \cap [0, 1]$ so that $B = A \cup C$, and A and C are disjoint. By additivity, $m(B) = m(A) + m(C)$, but we already know that $m(B) = 1$ and $m(C) = 0$, so it must be the case that $m(A) = 1$. Once again, it is worth noting that we can now readily compute the Lebesgue measure of sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ which gave us trouble when trying to integrate the Dirichlet function.

4 Hausdorff Measure

Although we will not formally present the extension of Lebesgue measure into higher dimensions, it is important to note that the Lebesgue measure, like it was the extension of length in $\mathbb{R}^1$, is the extension of area in $\mathbb{R}^2$, the extension of three-dimensional volume in $\mathbb{R}^3$, and so on. As a simple example, the two-dimensional Lebesgue measure of a rectangle defined by $[a, b] \times [c, d]$ would yield the area of the rectangle: $(b-a)(d-c)$. Likewise, the three-dimensional Lebesgue measure of a rectangular prism defined by $[a, b] \times [c, d] \times [e, f]$ would give us the volume of the prism: $(b - a)(d - c)(f - e)$. In general, one can easily imagine how Lebesgue measure extends, quite straightforwardly, into $\mathbb{R}^n$.

While this extension of Lebesgue measure is useful, it does not provide us with the complete mathematical machinery as we move into higher dimensions. For instance, when we are looking at a two-dimensional subset of $\mathbb{R}^2$, area is not the only quantity we are interested in. In the case of a rectangle, we might be interested in its side lengths or perimeter. In the case of a circle, it is useful to be able to compute its circumference. All of these quantities are one-dimensional measures of a subset in two-dimensions. Similarly, for a three-dimensional subset of $\mathbb{R}^3$, we could be interested in its surface area or diameter, in which case we need a two-dimensional measure as well as a one-dimensional measure of a three-dimensional subset.

In order to measure such quantities, we define a generalization of Lebesgue measure into higher dimensions which allow for the measure to be computed in different dimensions not necessarily corresponding to the dimension of the underlying set.

Definition 4.1. If U is a non-empty subset of $\mathbb{R}^n$, we define the **diameter** of U as $|U| = \sup\{|x-y| : x, y \in U\}$. If $A \subseteq \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a δ-**cover** of A.

As was in the case for Lebesgue measure in $\mathbb{R}^1$, our first step is to cover the set we are interested in with sets we can control. In $\mathbb{R}^1$, we simply used open intervals to accomplish this goal, but in order to measure subsets of $\mathbb{R}^n$ in various dimensions, we cannot simply use open balls in $\mathbb{R}^n$ which would inherently restrict the measure to dimension n. Consequently, we have defined the notion of a diameter, the largest one-dimensional distance between two points in a set, and we shall cover the set we are interested in, A, by taking the union of sets U_i where each of the U_i's have a diameter no larger than δ.

Definition 4.2. Let A be a subset of $\mathbb{R}^n$ and let $s \geq 0$. For $\delta > 0$, define

$$\mathscr{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } A\right\}.$$

Just as we took the infimum of the sum of lengths of the open intervals in the case of Lebesgue measure in $\mathbb{R}^1$, we are interested in the most efficient δ-covering of A such that the sum of the sth power of diameters is minimized. As you might suspect, s introduced in the definition above is the dimension we choose to measure the set A. Although we will not give a formal proof of this fact, it is relatively easy to show that $\mathscr{H}^s_\delta$ is an outer measure of $\mathbb{R}^n$ that satisfies the three conditions of Definition 2.1.

Definition 4.3. The **Hausdorff s-dimensional outer measure** of A is defined to be

$$\mathscr{H}^s(A) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(A) = \sup_{\delta > 0} \mathscr{H}^s_\delta(A).$$

As the name suggests, $\mathscr{H}^s$ also turns out to be an outer measure on $\mathbb{R}^n$.

Now, we restrict ourselves to only the subsets of $\mathbb{R}^n$ that are $\mathscr{H}^s$-measurable by way of Carathéodory's Criterion given generally in Definition 2.7.

Definition 4.4. A subset A of $\mathbb{R}^n$ is called $\mathscr{H}^s$-**measurable** if

$$\mathscr{H}^s(B) = \mathscr{H}^s(B \cap A) + \mathscr{H}^s(B \setminus A)$$

for all $B \subseteq \mathbb{R}^n$. The restriction of $\mathscr{H}^s$ to the σ-algebra of $\mathscr{H}^s$-measurable sets is called the **Hausdorff s-dimensional measure**.

Although we will not formally discuss this topic, it is worthwhile to note that the Hausdorff measure when we take $s = 1$ reduces to the Lebesgue measure in $\mathbb{R}^1$. By definition, if we have $A \subseteq \mathbb{R}$,

$$\begin{aligned}\mathscr{H}^1(A) &= \sup_{\delta>0} \mathscr{H}^1_\delta(A) \\ &= \sup_{\delta>0} \left\{ \inf \left\{ \sum_{i=1}^{\infty} |U_i| : \{U_i\} \text{ is a } \delta\text{-cover of } A \right\} \right\}.\end{aligned}$$

While the δ-covers $\{U_i\}$ in general do not have to be composed of intervals, we can see from our discussion in Appendix A that in $\mathbb{R}$, we cannot cover an interval more efficiently such that the outer measure adds up to less than its length. Therefore, the infimum of the summation of diameters for a δ-covering in $\mathbb{R}^1$ is the same as the infimum of the summation of lengths of open intervals that cover A. A similar relationship can be seen between Lebesgue measure in any higher dimension $\mathbb{R}^n$ and the Hausdorff measure when we take $s = n$, up to a constant scaling factor α_n. Thus, the Hausdorff measure is, indeed, a generalization of Lebesgue measure.

Example 4.5. *Let R be the rectangle $[a, b] \times [c, d]$ in $\mathbb{R}^2$.*

Let us compute the Hausdorff 1-dimensional measure of R. Given a certain $\delta > 0$, consider the δ-covering $\{U_i\}$ consisting of squares of side $\frac{\delta}{\sqrt{2}}$ such that the diameters (the diagonal of the squares) are equal to δ, satisfying our condition of a δ-covering. The area of each of these squares is $\frac{\delta^2}{2}$, and the area of R is $(b-a)(d-c)$, so we need at least $\frac{(b-a)(d-c)}{\delta^2/2} = \frac{2(b-a)(d-c)}{\delta^2}$ of these squares to cover R. Thus,

$$\begin{aligned}\mathscr{H}^1(R) &= \lim_{\delta\to0} \inf \left\{ \sum_{i=1}^{\infty} |U_i| \right\} \\ &\leq \lim_{\delta\to0} \frac{2(b-a)(d-c)}{\delta^2} \cdot \delta \\ &= \lim_{\delta\to0} \frac{2(b-a)(d-c)}{\delta} \\ &= \infty.\end{aligned}$$

This is not a formal proof that the Hausdorff 1-dimensional measure of a rectangle is equal to ∞. Note that in our computation, we have naively assumed that taking the limit over δ-coverings consisting of squares yields the same result as if we had considered all δ-coverings. This turns out to hold true for this specific case, and in general, the Hausdorff s-dimensional measure of a set that is D-dimensional where $s < D$ is always ∞.

Example 4.6. *Let L be the line segment $[a, b]$ on the x-axis in $\mathbb{R}^2$.*

Let us compute the Hausdorff 2-dimensional measure of L. Now, given a $\delta > 0$, consider the δ-covering $\{U_i\}$ consisting of intervals of length δ. Thus, the δ-covering requires at least $\frac{b-a}{\delta}$ of these intervals to cover L. Therefore,

$$
\begin{aligned}
\mathscr{H}^2(L) &= \lim_{\delta\to 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^2 \right\} \\
&\leq \lim_{\delta\to 0} \frac{(b-a)}{\delta} \cdot \delta^2 \\
&= \lim_{\delta\to 0} (b-a)\delta \\
&= 0.
\end{aligned}
$$

So the Hausdorff 2-dimensional measure of L must be equal to 0, and in general, the Hausdorff s-dimensional measure of a set that is D-dimensional where $s > D$ is always 0.

We have so far treated dimension according to our intuitive understanding, that is, to consider a set as 1-dimensional if it consists of line segments, 2-dimensional if it consists of areas, and so on. However, we can now extend the results illustrated by the previous two examples and give a precise definition of a dimension.

Definition 4.7. The **Hausdorff dimension** of a set A is the unique value $\dim(A)$ for which:

1. $\mathscr{H}^s(A) = \infty$ if $0 \leq s < \dim(A)$;
2. $\mathscr{H}^s(A) = 0$ if $\dim(A) < s < \infty$.

To see that there is such a unique value, first note that $\mathscr{H}^s(A)$ is non-increasing as s changes from 0 to ∞. Now, if $s < t$,

$$
\begin{aligned}
\mathscr{H}^s_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \right\} &= \inf \left\{ \sum_{i=1}^{\infty} |U_i|^t \cdot |U_i|^{s-t} \right\} \\
\geq \delta^{s-t} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^t \right\} &= \delta^{s-t} \mathscr{H}^t_\delta(A).
\end{aligned}
$$

Thus, if $\mathscr{H}^t(A) > 0$, the fact that $\lim_{\delta\to 0} \delta^{s-t} = \infty$ ensures that $\mathscr{H}^s(A) = \infty$. Similarly, if $\mathscr{H}^s(A) < \infty$, we see that $\mathscr{H}^t(A) = 0$. Therefore, there is a unique value $\dim(A) = \sup\{s : \mathscr{H}^s(A) = \infty\} = \inf\{s : \mathscr{H}^s(A) = 0\}$ which we call the Hausdorff dimension of A.

For most simple geometric shapes, the Hausdorff dimension is consistent with our intuitive understanding of dimension. For instance, the Hausdorff dimension of a line segment is 1, the Hausdorff dimension of a rectangle, circle, or triangle is 2, and the Hausdorff dimension of a cube, cylinder, or sphere is 3. However, notice that this definition does not stipulate that the Hausdorff dimension of a set must be an integer. In fact, there is a plethora of sets which have non-integral dimension. Interestingly, the rigorous mathematical definition of a "fractal", a term usually associated with complex geometric shapes that exhibit self-similarity, involves the Hausdorff dimension. As the name suggests, sets with fractional Hausdorff dimension, in general, are also fractals.

An example of such a fractal is the Cantor ternary set, $\mathcal{C}$, given in Example 3.14. Recall that the Cantor ternary set is the result of successively removing the middle third of each interval infinitely many times starting with the interval $[0, 1]$. Now, since we have discussed previously that the Lebesgue measure of the Cantor ternary set is 0, we know that its Hausdorff 1-dimensional measure is 0. However, this only asserts that $\dim(\mathcal{C}) \leq 1$, so it might be the case that we have measured the Cantor ternary set in the "wrong" dimension, a dimension other than its Hausdorff dimension. Indeed, this turns out to be the case.

Unfortunately, computing the Hausdorff dimension of fractals, in general, can be quite challenging, but we can make an educated guess for the Hausdorff dimension of the Cantor ternary set by exploiting its self-similarity. For self-similar sets, the dimension is the exponent that relates the magnification to the number of self-similar copies it produces. In other words, $N = r^D$ where r is the magnification, D is the dimension of the set, and N is the number of copies produced. This is illustrated for the 1-dimensional, 2-dimensional, and 3-dimensional cases in Figure 2. As you can see, for a line segment, magnifying it by 2 simply produces 2 copies of itself and magnifying it by 3 produces 3 copies of itself, and this follows the power-relationship $N = r^1$. Similarly, for a square, magnifying it by 2 produces 4 copies of itself and magnifying it by 3 produces 9 copies of itself, which follows the power-relationship $N = r^2$. A similar result is obtained for the three-dimensional cube. A complete discussion of this fact is outside the scope of this paper, so here we will simply accept the power relationship $N = r^D$ as truth.

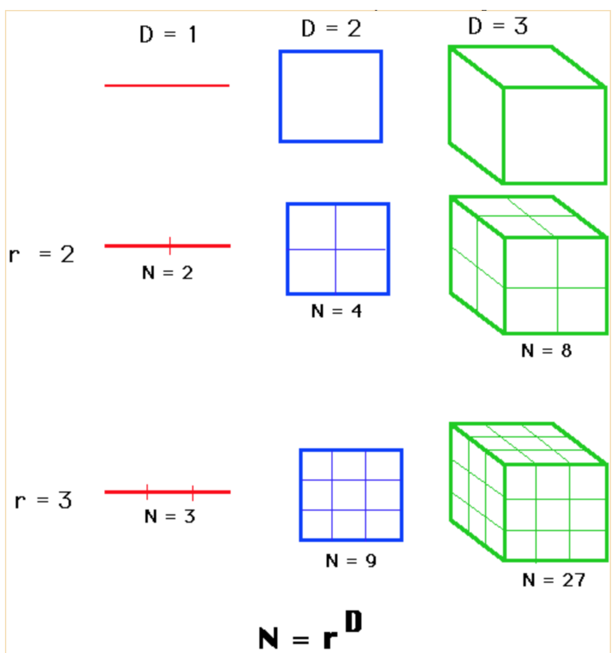

Figure 2: An Illustration of the Relationship Between Magnification, Dimension, and Number of Self-Similar Copies.

will replace image by creating my own

Now, examining the Cantor ternary set more closely, we notice that it is also a self-similar set. Applying the same argument as with the line segment, square, and cube, consider the magnification of the Cantor ternary set by 3. We immediately see that we have formed an exact copy of itself on the interval $[0, 1]$ and an additional copy of itself translated to the right by 2 to the interval $[2, 3]$. **provide another figure?** Essentially, the result is 2 copies of itself! Therefore, $N = 2$ when $r = 3$, and according to our power-relationship, the dimension of the Cantor ternary set should obey:

$$\begin{aligned} 2 &= 3^D \\ \implies \ln(2) &= D\ln(3) \\ \implies D &= \frac{\ln(2)}{\ln(3)} \end{aligned}$$

We will now verify this result and explicitly measure the Cantor ternary set in this dimension.

Theorem 4.8. *The Hausdorff dimension of the Cantor ternary set, $\mathcal{C}$, is $D = \ln(2)/\ln(3)$, and $\mathscr{H}^D(\mathcal{C}) = 1$.*

Proof. Let $D = \frac{\ln(2)}{\ln(3)}$. First, we will show that $\mathscr{H}^D(\mathcal{C}) \leq 1$.
Since $\mathcal{C} \subseteq C_n$, $\mathcal{C}$ can be covered by the 2^n intervals each of length 3^{-n} that form C_n. Thus,

$$\mathscr{H}^D_{3^{-n}}(\mathcal{C}) \leq 2^n(3^{-n})^D = 2^n(3^D)^{-n} = 2^n 2^{-n} = 1.$$

Now, as $n \to \infty$, $3^{-n} \to 0$, and we get the desired inequality $\mathscr{H}^D(\mathcal{C}) \leq 1$.
To show the opposite inequality, we will show that for any collection of intervals ϕ covering $\mathcal{C}$,

$$1 \leq \sum_{I \in \phi} |I|^D. \tag{1}$$

For each interval, we can include the endpoints to make it closed, and since $\mathcal{C}$ is also compact, we will only need to prove (1) for the case where ϕ is a finite collection of closed intervals. We can further reduce each interval $I \in \phi$ to the smallest interval containing some pair of disjoint intervals J and J' which occur in the construction of $\mathcal{C}$. Then, let K be the interval in the complement of $\mathcal{C}$ between J and J' such that the interval I is composed of J, K, and J'. Since the length of the open intervals removed is equal to the length of the closed intervals remaining at each stage of the construction of $\mathcal{C}$, we see that $|J| \leq |K|$ and $|J'| \leq |K|$. This inequality also holds for the average of $|J|$ and $|J'|$, so $\frac{|J|+|J'|}{2} \leq |K|$. Then,

$$\begin{aligned} |I|^D &= (|J| + |K| + |J'|)^D \\ &\geq \left(\frac{3}{2}(|J| + |J'|)\right)^D = 2\left(\frac{1}{2}(|J| + |J'|)\right)^D \\ &\geq 2\left(\frac{1}{2}|J|^D + \frac{1}{2}|J'|^D\right) = |J|^D + |J'|^D, \end{aligned}$$

where we have used the fact that $3^D = 2$ to get the second equality and the concavity of the function x^D to get the second inequality. Thus, we can replace the interval I by the two subintervals J and J' without increasing the sum in (1). We may recursively continue this reduction process until we have a covering of $\mathcal{C}$ with, say, equal intervals of length 3^{-n}. Hence,

$$\sum_{I \in \phi} |I|^D \geq 2^n (3^{-n})^D = 1.$$

Therefore, $\mathscr{H}^D(\mathcal{C}) = 1$, which also shows that $D = \frac{\ln(2)}{\ln(3)}$ is indeed the Hausdorff dimension of the Cantor ternary set. ■

A Appendix

proof to be added later